1977

A comparative analysis of various estimates of the hazard rate associated with the service life of industrial property

Ronald Eugene White
Iowa State University

77-26,019

WHITE, Ronald Eugene, 1941-
A COMPARATIVE ANALYSIS OF VARIOUS ESTIMATES OF THE HAZARD RATE ASSOCIATED WITH THE SERVICE LIFE OF INDUSTRIAL PROPERTY.

Iowa State University, Ph.D., 1977
Engineering, industrial

Xerox University Microfilms, Ann Arbor, Michigan 48106

A comparative analysis of various estimates of the hazard rate associated with the service life of industrial property

by

Ronald Eugene White

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department: Industrial Engineering
Major: Engineering Valuation

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1977

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INTRODUCTION

The motivation for this study stems from an interest in the quantitative methods used by depreciation engineers to estimate the probable service life of industrial property. While it is now generally understood that an accurate estimate of service life is indispensible to the application of most methods of computing depreciation, this was not always the case. A variety of schemes were once used to charge the cost of depreciable plant to expense without regard to the service life of the property. Under the retirement method, for example, the cost of a plant asset was first charged to a plant account and then charged to expense at the time of retirement. An alternative treatment that supposedly kept the property in 100% operating condition was to charge a plant account with the cost of the original plant asset, but replacements were charged to expense. Since depreciation methods such as these did not aim to distribute the cost of an asset over its productive life, there was little need for making engineering estimates of the probable service life.

Shortly after the turn of the twentieth century these earlier methods were gradually abandoned and full depreciation accounting became the accepted method of charging the cost of depreciable plant to expense. The usual accounting practice today in industries using long-lived assets is to allocate a portion of the investment in depreciable plant to each accounting period during the life of the plant. Thus, the cost of the depreciable property that is charged to a fixed asset account is viewed as a prepaid expense to be amortized over the accounting periods related to its use.

The Interstate Commerce Commission (ICC) played an important role in promoting acceptance of the allocation of cost concept and the age-life relationship in depreciation accounting. As early as 1907, full depreciation accounting was prescribed by the ICC for steam railroads. In 1910, the jurisdiction of the ICC was broadened to include telephone, telegraph, and cable companies that were engaged in interstate message communications. Shortly thereafter, accountants of the ICC began work on a Uniform System of Accounts for Telephone Companies that included definitions and rules for depreciation accounting. It was not until January 1, 1913, however, that the work was completed and the accounting system became mandatory. According to the Depreciation Subcommittee of the National Association of Regulatory Utility Commissioners (52, p. 10), the rules with respect to depreciation included the following statement:

> ". . . depreciation expense should be designed to recover the cost of plant over its estimated life in the case of individual units, and over the estimated average service life in the case of group properties."

It is reported by Nash (49, pp. 4-5) that a more comprehensive accounting system was adopted by the ICC in 1914, wherein the program with respect to depreciation was defined as follows:

> "We therefore find that annual depreciation charges shall be computed at such percentage rates of the ledger value of the unit of property in question that the service value, as hereinbefore defined, may be distributed under the straight-line method in equal annual charges to operating expenses during the estimated service life of the unit. Annual charges so computed shall be reduced to a monthly basis by dividing by 12."

Thus, the introduction in 1913 of the Uniform System of Accountants required by the ICC under the Mann-Elkins Act of 1910, firmly established the propriety of depreciation accounting which, in turn, created a general need for the development of sound methods of estimating the probable service life of industrial property.

To the uninitiated it might seem that an estimate of probable service life could be obtained by merely calculating the average age of plant retired in recent years. But with a little reflection it becomes apparent that the problem is not this simple, since the average age of plant retired is typically lower than the true average service life. For example, if we calculate the average age at death of the male population born in 1920 who have already passed on, we will surely understate the average life of all males born in 1920, since it is reasonable to assume that a majority of them are still alive.

With the exception of short lived property such as motor vehicles, office furniture, and communication equipment, most classes of industrial property have not been in service long enough to provide a history of completed generations. Consequently, it is necessary to devise methods of estimating probable service life from a series of vintages that are only partially retired. The problem is further complicated by the fact that many firms do not maintain plant accounting records that reveal the age distribution of plant still in service. In this case, estimates of the probable service life must be derived without any knowledge of the age of plant retirements at the time of their retirement.

While most of the common methods of computing depreciation require an estimate of service life, some methods also require an estimate of life expectancy which is the period of time extending from an observation age to the forecasted date of retirement. This information, which is also needed for depreciation reserve studies, can be obtained from a mathematical formulation of the life characteristics of the property under review. The mathematical expressions used to describe these characteristics are known as "survival functions" which are derived by the depreciation engineer from the application of various life analysis techniques.[1]

The purpose of the present study is to investigate the possibility of improving the estimation procedure currently employed in the application of a sub-set of the class of life analysis techniques known as the actuarial methods. This investigation will focus on the annual rate (or retirement rate) method of life analysis and the statistic used to estimate the hazard rate for each age-interval.

[1]The term "life analysis" has traditionally been used by depreciation engineers to describe the application of certain analytical procedures to plant accounting records containing the life history of various classes of physical property. The end result of such an analysis is a mathematical description of the age distribution of plant retirements measured in units of realized service. The term "life estimation" is also used by the depreciation engineer when attention is given to predicting the expected remaining service life of property units still exposed to the forces of retirement. The two terms are not synonymous; life analysis is concerned with history and life estimation is concerned with the future. The present study is limited to a consideration of life analysis.

RELATED CONCEPTS

Once the need for service life estimates had been established, it was soon recognized that such estimates could be obtained by applying the actuarial procedures developed for investigating human mortality to the mortality experience of physical property. But these procedures (used extensively in life insurance work) can only be applied to plant accounting records that reveal the age of a plant asset at the time of its retirement. In other words, each property unit must be identifiable by date of installation and age at retirement. This limitation encouraged the development of a class of life analysis techniques known as the "semi-actuarial" methods.

Semi-actuarial Methods of Life Analysis

In 1922, Cyrus G. Hill (34) proposed a method for analyzing the life experience of various classes of telephone plant when ". . . the age of the plant retired at any time cannot be told from a casual inspection of the books." In other words, the available property records reveal the annual gross additions and annual plant or account balances (i.e., plant in service) with no indication of the age of plant retirements.

The Hill method is a trial and error procedure that attempts to duplicate the most recent plant balance of a plant account by distributing the annual gross plant additions over time according to an assumed life table or survivorship function. The constructed or computed plant balance is simply the accumulation of each gross plant addition multiplied by the indicated proportion surviving (from the assumed life table) at its attained age. If the mortality experience of the property had, in fact, followed the life characteristics described by the assumed survivorship function then the computed plant balance would be equal in magnitude to the amount of plant actually in service. On the other hand, if the selected survivorship function does not generate adequate retirements (i.e., the computed balance is greater (less) than the actual balance), then the procedure would be repeated using a shorter (longer) average service life with a survivorship function of the same dispersion.

An obvious drawback in Hill's method is that every survivorship function has an average service life that will produce a single computed balance equal in magnitude to the corresponding actual plant balance. Furthermore, since the derived average service life is a function of the selected dispersion, an incorrect dispersion will introduce an error in the estimated average service life.

In 1943, a variation of the Hill method was presented by the National Association of Railroad and Utilities Commissioners (50) in a report of the committee on depreciation. While the principle of the suggested procedure (described as the "Indicated Survivors Method") is identical to Hill's, the NARUC method attempts to duplicate a series of plant balances over a few prior years instead of limiting the analysis to the most recent plant balance in the account. The advantage gained from the use of multiple balances is that it may provide a clue to the probable type of dispersion. The claimed advantage is questionable, however, since the selection criterion is simply a visual inspection of how well the series of computed balances conforms to the series of actual balances.

In 1947, Alex E. Bauhan (5) presented a paper at the American Gas Association-Edison Electric Institute National Accounting Conference that described a method for analyzing mass property accounts (i.e., aged retirements are not available) that would provide an estimate of both dispersion-type and average service life. The Bauhan procedure (known as the "Simulated Plant Balances Method") is a variation of the Indicated Survivors Method that incorporates a minimum sum of squares criterion in the selection of an appropriate dispersion.

At the same conference, Henry R. Whiton (63) and Paul H. Jeynes (37) each presented papers that outlined two additional procedures for estimating dispersion-type and average service life from mass mortality data. In brief, the Whiton method suggested matching cumulative retirements and the Jeynes method suggested matching annual retirements derived from a record of annual net additions and a theoretical renewals function. These two methods have been named the "Simulated Plant Cumulative Retirements Method" and the "Simulated Plant Indicated Renewals Method", respectively.

A more recent development that has attracted a certain amount of attention is the "Simulated Plant Period Retirements Method". This procedure was originally suggested by William D. Garland (24) in a paper presented at the 1968, American Gas Association-Edison Electric Institute National Accounting Conference. Unlike the earlier methods, Garland's approach develops a "best-fitting" average service life for a selected survivorship function by seeking a sum of differences between actual and computed retirements approximating zero over a specified time period.[1] Although the Period Retirements Method is a relatively new innovation, it

[1] An earlier version of the Period Retirements Method was presented by Garland (23) at the 1967, A.G.A.-EEI National Accounting Conference. The earlier version used a minimum sum of squares criterion.

and the Balances Method are probably the most widely used of the above techniques.

In view of the apparent similarity in the methods just described, they have become known (collectively) as the "Simulated Plant-Record" or "SPR" method. As this name implies, the SPR method is simply a trial and error procedure that attempts to duplicate (i.e., simulate) some portion of a plant accounting record that may or may not permit age identification of plant retirements. The method, however, is usually associated with mass mortality data.

Actuarial Methods of Life Analysis

The actuarial methods of life analysis differ from the semi-actuarial methods in two important respects. First, the actuarial methods require plant accounting records that provide complete age identification of current and past retirement experience; each unit of property must be identifiable by date of installation and age at retirement. Secondly, the actuarial methods are not a trial and error procedure; they are a procedure that involves two distinct steps, both of which can be approached in several different ways.

The first step involves a systematic treatment of the available data for the purpose of constructing a life table.[1] The theory and application of the life table is a well-known topic in the field of statistics. It has many applications in various areas of research where birth, death, and illness may take place. According to Chiang (11), the earliest life

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[1]The format of a life table is given in Table 1, p. 32.

tables date as far back as the seventeenth century; Halley's life table for the City of Breslau, published in the year 1693, apparently contained most of the columns in use today. The subject matter, however, is by no means limited to human mortality. Zoologists, biologists, physicists, engineers, and investigators in other fields have found the life table a valuable means of presenting mortality data.

The construction of a life table for depreciation applications usually involves one of at least five available methods. Winfrey (66, pp. 17-18) describes these as: the individual-unit method; the original-group method; the composite original-group method; the multiple original-group method; and the annual-rate method. Of these five methods, only the annual-rate method will produce a complete life table. The other methods produce an abbreviated table (i.e., one that does not contain all of the columns normally associated with a life table) that minimally contains an estimate of the cumulative proportion surviving.

The individual-unit method is the least sophisticated of all the methods since it only considers units that have been retired from service; it does not give any weight to the units remaining in service at any given age. The cumulative proportion surviving is obtained by arranging the retirements during a given year or series of years in ascending order according to the age of each unit at its retirement. The sum of all such retirements is taken as an estimate of the units exposed to retirement at age zero. The number of units subject to retirement at the beginning of each successive age-interval is easily obtained by subtraction and the ratio of these exposures to the sum of all retirements provides an estimate of the cumulative proportion surviving. Unlike the other methods, this method will always produce a life table extending to zero percent (or proportion) surviving at maximum life.

The original-group method of constructing a life table gives weight to both the retirements and survivors of the property units installed as a group or vintage in a given calendar year. The method does not consider more than a single vintage, however, which will result in a censored life table (i.e., non-zero percent surviving in the last tabulated age-interval) if the original group is not fully retired. Clearly, the ratio obtained by dividing the number of units installed at age zero into the number of units surviving at the beginning of each successive age-interval will generate the cumulative proportion surviving.

The composite original-group method is a variation of the original-group method that can be used when the number of units in a single vintage is deficient or the cumulative proportion surviving is extremely erratic. The method simply combines the retirements and survivors of equal ages from two or more vintages into a composite group which is treated as a single original-group. Thus, the cumulative proportion surviving is calculated on the basis of the combined total of the survivors from all vintages included in the composite group.

The multiple original-group method is also a variation of the original-group method wherein the cumulative proportion surviving at each age-interval is obtained from a different vintage. Thus, while the original-group method considers a single vintage over a series of observation dates, the multiple original-group method considers a series of vintages at a single observation date. Estimates of the cumulative proportion surviving that are obtained using this method are typically irregular because successive vintages seldom exhibit an equal proportion surviving at equal ages. This is not a serious problem, however, since most life tables require some form of graduation.

The annual-rate method is the most sophisticated of the five methods under review and will be used in this study to construct the observed life table. The mechanics of the annual-rate method require the calculation of a series of ratios obtained by dividing the number of units surviving at the beginning of an age-interval into the number of units retired during the same interval. This important ratio (or set of ratios) is variously known as the hazard rate, the rate of mortality, the force of mortality, the conditional proportion retired, the retirement rate, or the retirement ratio. Having calculated this ratio for each age-interval, the cumulative proportion surviving is obtained by multiplying the conditional proportion retired for each age-interval by the proportion surviving at the beginning of that age-interval and subtracting the product from the proportion surviving at the beginning of the same interval. The annual-rate method can also be applied to multiple vintages by combining the retirements and/or survivors of like ages from each of the vintages included in the analysis. The data selected under either the composite original-group method or the annual-rate method may be for a specified "additions era" or for a specified "retirements era". The use of an additions era means that the analysis is restricted to the record of retirements and survivors from plant added during the years included in the selected era. The use of a retirements era means that the analysis is restricted to the retirement activity of all vintages represented by survivors at the beginning of the selected era.

The construction of a life table by any of the above methods has been identified as the first step in applying the actuarial methods of life analysis. The second step involves graduating the observed life table and fitting the smoothed series to a family of survival functions. The functions used are either empirically derived or otherwise known to be representative of the mortality characteristics encountered in the field of study in which they are being applied.

Graduation of an observed life table can be justified from both a theoretical and a practical point of view. According to the mathematical theory of probability, the irregularities observed in a life table of physical property can be attributed to errors of observation or chance fluctuations that arise because of the limited and necessarily finite extent of the observations. If it were possible to secure unlimited data, it is believed that the irregularities would become insignificant. Thus, the process of graduation can be viewed as a technique for estimating the series of true rates of mortality that is assumed to have given rise to the irregular series of observed probabilities.

As a practical matter, life tables of physical property often contain irregularities due to events that are unlikely to occur again at the same ages or at the same relative frequency. A major accident, for example, or a management decision to retire a certain class of property can produce irregular variations in a life table that are not representative of the underlying forces of mortality. Thus, the graduation process is frequently used to remove irregularities which the depreciation engineer has reason to believe are not a feature of the true, underlying rates of mortality. Graduation techniques are also used to extend a censored life table to zero percent surviving. A censored life table must be extended before the probable average service life can be computed.

Several methods have been developed to graduate an observed series. These methods are classified by Miller (48) as follows:

(i) The graphic method. In this method, the observed values are suitably plotted on graph paper and among them a smooth, continuous curve is drawn as the basis of the graduated series.

(ii) The interpolation method. In this method, the data are combined into age groups and the graduated series is obtained by interpolation between points determined as representative of the groups.

(iii) The adjusted-average method. In this method, each term of the graduated series is a weighted average of a fixed number of terms of the observed series to which it is central.

(iv) The difference-equation method. In this method, the graduated series is determined by a difference equation derived from an analytic measure of the relative emphasis to be placed upon fit and smoothness.

(v) Graduation by mathematical formula. In this method, the graduated series is represented by a mathematical curve fitted to the data.

Of these methods, the graphic approach and graduation by mathematical formula are the most widely used in the field of depreciation. The graphic method is usually applied to the cumulative proportion surviving and may or may not involve the use of standard curves, such as the Iowa-Type survivor curves. If the observed data are sufficiently smooth and not extremely censored, a freehand curve can be drawn among the plotted points that will be satisfactory for most applications. The use of type or standard curves offers a refinement to the graphic method that removes much of the subjectivity that is inherent in drawing a freehand curve.

The standard curves developed by Kurtz (44) and Winfrey (66) at the Iowa Engineering Experiment Station (now known as the Engineering Research Institute) are, by far, the most widely used. These so-called Iowa-Type Curves were originally presented in Bulletin 103 (67) as a set of 13 generalized retirement frequency curves that were obtained from an analysis of the retirement experience of 65 property groups.[1] The original set of 13 curves was later modified slightly and expanded to include 5 additional curves that were developed by Winfrey (66) from an analysis of 124 property groups which included the 65 groups contained in the earlier study. The Iowa-Type Curves now number 22 which includes 4 origin-moded curves developed by Couch (14).

The Iowa Curves are mathematically described in terms of the Pearson frequency curve family and are classified according to the location of the mode of the retirement frequency curve relative to the average life as well as the maximum height of the modal ordinate. The set now includes seven symmetrical, five right-modal, six left-modal, and four origin-modal curves. The mathematical form of the symmetrical frequency curves is given by

$$y = y_0(1 - \frac{t^2}{a^2})^m$$

[1]The first 52 property groups contained in this study were grouped initially into 7 type curves and published by Kurtz (44) in 1930.

which is a Pearson type II. The constants in this equation are y_0, a, and m. The variable t represents age (in units equal to 10 percent of average life) measured from the average life ordinate. The right-modal and left-modal curves were obtained by separating the observed frequencies into a major and a minor constituent curve, each of which was fitted to a Pearson type I and summed to obtain the total frequency. The resulting curves are described by a general equation of the form

$$y = Y_e(1 + \frac{t}{A_1})^{M_1}(1 - \frac{t}{A_2})^{M_2} + y_e(1 + \frac{t}{a_1})^{m_1}(1 - \frac{t}{a_2})^{m_2}$$

where Y_e, A_1, A_2, M_1, M_2, y_e, a_1, a_2, m_1, and m_2 are constants. The origin-modal curves (except for the group classified as 0_1) were obtained through trial and error adjustment of a Pearson type VIII curve which is given by the general equation

$$y = y_0(1 + \frac{t}{a})^{-m}.$$

The group classified as type 0_1 are represented by a straight line having an ordinate value of 5.0 for all values of t between -10 and +10.

Since the cumulative proportion surviving is the most common and convenient series to graduate using the graphic method, the Iowa-Type Curves were numerically integrated to produce equivalent survivor curves that have been drawn on sheets of graph paper to an appropriate scale. Thus, an observed series is easily graduated by plotting the cumulative proportion surviving on a sheet of transparent graph paper and overlaying each sheet of survivor curves with the sheet of plotted data. The type curve and average life which best fit the data are determined by visual inspection. This procedure has also been computerized using a minimum sum of squares or a minimum algebraic sum of the differences between the data points and the fitted curve as the selection criterion.

The Iowa-Type Curves are not, however, the only type curves available for life studies of industrial property. In 1947, Kimball (42) introduced the so-called h-System which was formulated by Gumbel (31) in 1933 as a system of survival functions for human mortality. Unlike the Iowa Curves which were empirically derived from an analysis of actual retirement data, the h-System is described by a single mathematical function that is derived from a theoretical consideration of the parametric form of a truncated normal probability distribution.[1] The resulting retirement frequency curves are left-moded, however, which has possibly discouraged a more widespread use of the system.

Depreciation personnel of the Bell Telephone System have, for many years, used the so-called Gompertz-Makeham formula to graduate an observed life table. This formula was also developed from life studies of human mortality and later applied to the retirement experience of physical property. It is reported by Jordan (38) that in 1825, Benjamin Gompertz, in a celebrated actuarial paper, examined the effect of assuming "the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals." In other words, Gompertz assumed that man's power to

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[1]A complete derivation of the h-System is contained in Appendix A.

resist death decreases at a rate proportional to itself, which is equivalent to the assumption that the force of mortality increases in geometric progression. This can be stated mathematically by letting

$$\lambda(t) = Bc^{t}$$

where $\lambda(t)$ is the hazard function, B and c are constants, and t is age measured in units of time. Gompertz's expression for the survivorship function can be derived using the well-known functional relationship between the hazard function and the survivorship function.[1] Thus, if

$$\Lambda(t) = \int_0^t \lambda(x)\,dx = \int_0^t Bc^{x}\,dx = \frac{B}{\ln c}(c^{t} - 1)$$

$$= -(c^{t} - 1)\ln g = -\ln g^{c^{t}-1}$$

where $\ln g = -\frac{B}{\ln c}$, then the survivorship function S(t) is

$$S(t) = e^{-\Lambda(t)} = e^{\ln g^{c^{t}-1}} = g^{c^{t}-1} .$$

This expression, however, is usually written as

$$S(t) = kg^{c^{t}}$$

where $k = 1/g$.

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[1] Infra, p. 38.

In presenting his formula, Gompertz, as quoted by Jordan (38, p. 25), stated:

> "It is possible that death may be the consequence of two generally coexisting causes: the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or increased inability to withstand destruction."

In deriving his law of mortality, however, Gompertz considered only the second of these causes. In the year 1860, William Makeham combined the two causes in a formula that turned out to be a remarkable improvement on Gompertz's assumption. The effect of the first cause, chance, would be the addition of a constant term to the Gompertz hazard function. Hence, Makeham's assumption may be written as

$$\lambda(t) = A + Bc^{t} .$$

Makeham's expression for the survivorship function can be derived in the same manner as the Gompertz expression. Thus, if

$$\Lambda(t) = \int_0^t \lambda(x)\, dx = \int_0^t (A + Bc^{x})\, dx$$

$$= At + \frac{B}{\ln c}(c^{t} - 1) = -\ln s^{t} - \ln g^{c^{t}-1}$$

where $\ln x = -A$ and $\ln g = -\frac{B}{\ln c}$, then the survivorship function S(t) is

$$S(t) = e^{-\Lambda(t)} = e^{\ln s^{t} + \ln g^{c^{t}-1}} = s^{t} g^{c^{t}-1} .$$

Again, this expression is usually written as

$$S(t) = ks^t g^{c^t}$$

where $k = 1/g$.

Makeham's contribution did not, however, detract from the usefulness of Gompertz's formula; both of these laws possess properties that are desirable for practical applications. Gompertz's law was employed in the construction of the 1937 Standard Annuity Table, and Makeham's law was used in connection with the Commissioners Standard Ordinary Mortality Table and also with the 1949 Annuity Table.

It is not known exactly how these laws came to be used by those working with life analysis of industrial property.[1] But at some point in time, the Makeham law, as it is called by actuaries, was renamed the Gompertz-Makeham formula by those in the life analysis field. Presumably, this dual reference was intended to give credit to both authors.

Since each of these formulas contains a number of unspecified parameters, each gives rise to an infinite number of different survival functions. These laws of mortality thus define only the form of the mathematical functions to be assumed and do not yield numerical measurements of mortality until appropriate values are chosen for the parameters. Although both Gompertz's and Makeham's laws appear well-suited to life insurance applications, several researchers including Winfrey (66, p. 40) have found that neither the Gompertz formula nor the Makeham formula

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[1]Winfrey (66, p. 8) reports that to his knowledge, the first printed reference to the use of the Gompertz-Makeham formula in dealing with retirement data of physical property was in testimony presented in 1928 by the American Telephone and Telegraph Company before the Interstate Commerce Commission in Docket No. 14,700.

expresses a totally satisfactory mathematical law for industrial properties.

In the early 1930's, Lawrence S. Patterson, of the New York State Public Service Commission, developed a system of generalized survival functions that became known as the Patterson System (42). The mathematical form of the survivorship function described by this system is given by

$$\begin{aligned} S(t) &= 1 - t^n/2, & 0 \le t \le 1, \\ &= (2 - t^n)/2, & 1 \le t \le 2, \end{aligned}$$

where t denotes the age in percent of average service life, and n is a parameter to be determined. The variance of the generalized retirement frequency curve of the above system (with average service life equal unity) can be shown to be

$$\sigma^2 = 2/[(n + 1)(n + 2)] .$$

Thus, the Patterson System represents a two-parameter family of survivorship functions, with the average service life acting implicitly as one parameter, and the index n determined by the variance σ^2 of the generalized retirement frequency curve, serving as the second parameter. According to Kimball (42), the Patterson System is oversimplified for some purposes, but has been found useful for turnover-cycle computations. This is not surprising, however, since all of the retirement frequency distributions contained in this system are symmetrical.

While each of the above type curve systems is adaptable to the graphic method of graduation, some may also be used in the process of graduation by mathematical formula. In the formula method of graduation, the graduated series is represented by a mathematical function fitted to the data. The application of the method involves two steps:

(i) the choice of the form of function to represent the graduated series; and

(ii) the estimation of the parameters of the chosen function.

Mathematical functions chosen for this purpose are usually continuous, differentiable, and involve relatively few parameters. A second or third degree polynomial, the normal probability distribution, and the Gompertz formula are examples of such functions. While tests applicable to the data are sometimes helpful, the selection of an appropriate function is largely a matter of experience. The parameters of the chosen function are usually estimated by the method of moments, least squares, maximum likelihood, or some variation of them.

The formula method of graduation can be used to smooth and extend either the observed retirement frequency distribution, the conditional proportion retired, or the cumulative proportion surviving. The process of graduating an observed retirement frequency distribution by formula is essentially the problem considered by Kurtz and Winfrey in the development of the Iowa-Type Curves. Their investigation, as noted earlier, resulted in selecting the Pearson frequency curve family to represent the graduated series, while the parameters of the chosen function were estimated by the method of moments. Winfrey (66) later investigated the Gram-Charlier series as an alternative to the Pearsonian system, using both the method of moments and the method of least squares to estimate the parameters of the series. It is reported by Winfrey (66, p. 76), however, that ". . . the author had little success in getting a direct fit with (the Gram-Charlier series) except for the symmetrical frequency curves."

A related approach to the problem of frequency graduation is discussed by Buehler (9) who offers a formula for estimating the parameters β_i of a function $\phi(x) = \beta_1\phi_1(x) + \ldots + \beta_m\phi_m(x)$ in such a way that ϕ has approximately some specified distribution $g(\phi)$ which, for example, could be a normal distribution. This approach is based on the work of Hammersley and Morton (32) who investigated the function $\phi(x) = \alpha + \beta x$ as a transformation of observed values x grouped in a frequency distribution. Although Krane (43) draws freely on Buehler's method in working with the hazard function, this author is not aware of any research in the field of life analysis that has used Buehler's technique to graduate a retirement frequency distribution.

The Gompertz-Makeham formula is the function most often chosen to represent a graduated series of the cumulative proportion surviving. There are differences of opinion, however, as to the merits of graduating this series vis-a-vis the retirement frequency distribution or the conditional proportion retired. Benson (in Ref. 51, p. 78), for example, is opposed to mathematically graduating either the cumulative proportion surviving or the retirement frequency distribution for the following reasons:

> "The Gompertz-Makeham equation used by life insurance actuaries and the modified Gompertz-Makeham equation used by the Bell System Companies are open to the serious objection that the manipulative treatment of the data by the successive multiplication of 'observed' survival ratios to obtain an 'observed' life table, before the fitting process can be begun, destroys to a large extent the independence of the individual observations.

Furthermore, the necessity of having to assume an end point and, in many cases, a value for the negative logarithmic differential at age 0 requires the introduction of judgment at an early stage in the process. This is especially objectionable when the data end considerably short of the ultimate limit of life."

"The Kurtz method of fitting Pearsonian frequency curves to retirements computed from 'observed' life tables uses data even further removed from independence than does the Gompertz-Makeham method."

In defense of its practice, the Bell Telephone System (2, Chapter 2, p. 31) has taken the following position:

". . . it is sometimes suggested that, before graduating, the Depreciation Engineer should plot the observed survival rates (or the mortality rates which are the complements thereof) and graduate them, first changing or relocating any points which seem to be out of line. Otherwise, so the argument goes, unless this is done, the entire remaining portion of the observed life table could be thrown out of line because of some unusual happening in a single age interval. The Bell System position on the other hand is that the future life characteristic . . . can best be estimated with <u>actual</u> past experience as a guide. To the extent that this past experience was unusual, the Depreciation Engineer can temper his estimates accordingly. But obviously he needs to know what it actually was regardless of whether, or to what extent, it appeared to be abnormal. Otherwise, he would be hopelessly misled by a series of 'normalized' life indications."

The Depreciation Committee of the American Gas Association and the Depreciation Accounting Committee of the Edison Electric Institute (1, p. 40) have (perhaps wisely) avoided the controversy by taking the following stand:

"In passing it may be noted that in the past there has been some spirited controversy over the contention by some analysts that the fitting of a smooth curve to retirement ratios was superior to fitting the percent survivor stub curve. The consensus at the present time is that neither is superior to the other. One can sometimes obtain quite different mortality curves by these two methods - from the same set of data."

The National Association of Regulatory Utility Commissioners (52, p. 117) has summarized most of the arguments advanced in favor of graduating the conditional proportion retired as an intermediate step in the process of obtaining a smooth survivorship function. According to the Association, the advocates of this method contend:

(i) that retirement ratios are the most independent since they are nearest to the raw data;

(ii) that the retirement ratio at one age need not necessarily influence those at other ages, as contrasted with the chain relationship of the retirement frequency distribution or the cumulative proportion surviving where each element of the series depends on all those which have gone before;

(iii) that no fundamental law of mortality characteristics need be assumed beyond that of the elementary one that the older property is, the more likely it is to be retired.

(iv) that experience has shown that a simple type of equation can be used to describe the retirement ratio curve, and that therefore the data can be allowed to dictate the form of this equation; and

(v) that consequently the mathematical procedure is simpler than in the other actuarial methods.

The function most often chosen to represent a graduated series of the conditional proportion retired is a polynomial of the form

$$\lambda(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \ldots + \beta_n t^n .$$

Experience has shown, however, that it is rarely necessary to use a polynomial of greater than third degree (52, p. 118). The parameters of this function are usually estimated by the method of least squares or by Fisher's adaptation of the orthogonal polynomials of Tchebycheff (50, p. 248).

One of the less obvious advantages to be gained from graduating the conditional proportion retired stems from an important statistical property of the data. It is well-known (68, p. 95) that the variance of the conditional proportion retired is different for each age-interval, which suggests estimating the parameters of the assumed hazard function by weighted least squares. A potential difficulty, however, is that estimates of the hazard function are based on observed conditional probabilities and there is clearly some correlation among these since the survivors of the k^{th} age-interval constitute the sample size for the $(k+1)^{st}$ age-interval. But it has been shown by Chiang (11) that the covariance between the conditional proportion retired in two age-intervals is asymptotically zero which, at least in large samples, eliminates the need for estimating parameters by a generalized least squares approach. This property has allowed several researchers, including Henderson (33) and Lamp (45), to investigate various methods of weighting that reflect serial independence of the disturbance term. It should be noted, however, that zero covariance between the conditional proportion retired in two age-intervals does not establish their independence. In fact, it can be shown and has by Chiang (11) that the conditional proportion retired (or conditional proportion surviving) for two non-overlapping age-intervals are not independently distributed.

While some attention has been given to methods of weighting, this author is not aware of any research in the field of life analysis that has considered the problem of selecting the best estimator of the hazard rate for each age-interval to be used in estimating the parameters of an assumed hazard function. A logical choice is, of course, the observed conditional proportion retired, which is the estimator associated with the annual-rate method of constructing a life table. Other estimators can be derived, however, that may be superior to the conditional proportion retired. This study will undertake such an investigation which, hopefully, will lead to a better understanding of the mortality characteristics of industrial property.

STATEMENT OF OBJECTIVES

It was stated earlier that graduation by mathematical formula generally involves two steps:

(i) the choice of the form of function to represent the graduated series; and

(ii) the estimation of the parameters of the chosen function.

This study is primarily concerned with step two which is quantitative in nature and well-suited to empirical investigation. Step one is an equally important consideration in the process of life analysis, but it is far more subjective since there is no known function that expresses a totally satisfactory mathematical description of all of the forces of retirement. This does not, however, detract from the importance of step two; the procedure used to estimate the parameters of the chosen function should be statistically sound regardless of the form of the selected function.

The procedure used to estimate the parameters of a hazard function in life studies of industrial property has traditionally relied on the conditional proportion retired as an estimate of the hazard rate for each age-interval. This is a logical choice, however, since the conditional proportion retired is an estimate of the probability of retirement during an age-interval, conditioned upon exposure to the risk or forces of retirement at the start of the interval. Intuition, experience, and research have also led to various methods of weighting the conditional proportion retired from which the parameters of an assumed hazard function are usually estimated by the method of least squares.

A review of the literature in other fields of investigation reveals that few, if any, researchers using the methods of actuarial statistics rely on the conditional proportion retired (or dying) as an estimate of the hazard function. The statistic most often used in the biomedical sciences is the so-called actuarial estimate which is obtained by dividing the average number of survivors over a given age-interval into the number of retirements during the interval. The parameters of an assumed hazard function are then estimated by the method of least squares. It was also found that researchers in radiology have used the colog of the survivor ratio as an estimate of the hazard function. This statistic, which can be shown to be the maximum likelihood estimate of the hazard rate, is also used by actuaries in the development of annuity benefits. Thus, the fact that other researchers have rejected the conditional proportion retired suggests that it may not be the best estimate of the hazard rate for depreciation applications.

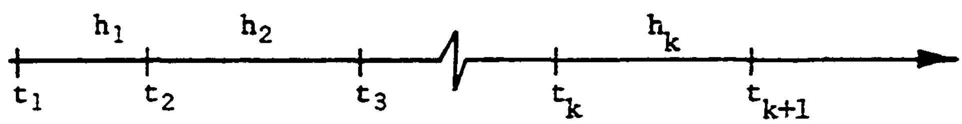
The objective of this study is to derive and compare various estimates of the hazard rate (or hazard function) associated with the service life of industrial property and determine which, if any, is best for depreciation applications. The term "best" as used in this study is taken to mean an estimate of the hazard rate that consistently yields estimates of the parameters of an assumed hazard function that are closest to the true, underlying population parameters.

MATHEMATICAL DESCRIPTION OF THE DATA

This section provides a mathematical description of the life table and a development of the probability relationships defined by the survival functions. The notation and functional relationships introduced in this section will be used in the next section to derive estimates of the hazard rate which, in turn, will be used to obtain estimates of the parameters of a hazard function.

The Life Table

Consider the following time axis where t_k represents a discrete point in time and h_k denotes an interval of time between points t_k and t_{k+1}:



In depreciation applications, h_k is called an "age-interval" and is measured from the beginning of one period of observation to the beginning of the next consecutive period. For practical reasons, it will be assumed that observations are made on December 31 such that a property unit or group of property units installed at time t_1 will have attained an age of t_k years at the k^{th} observation date.[1] We will also assume that plant additions and retirements are distributed uniformly throughout the year

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[1]The measurement of rendered service in time units of a year is arbitrary. A unit of time less than a year (month, week, day) or units of production (pounds, cubic feet, gallons) could be employed with no loss of generality. The year has been adopted as a unit of measurement by virtue of its conformity to the standard accounting interval used in depreciation calculations.

such that the average age of plant in service at the end of the year in which it was installed is one-half year. This assumption (which is equivalent to assuming that all plant additions are made on July 1) is known in the field of depreciation as the "half-year convention". By definition, therefore, the domain of t_k is restricted to the set of numbers $\{0, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots\}$. Thus, an age-interval can be specified either by reference to its end points (i.e., $t_k - t_{k+1}$) or by reference to its position relative to age zero which is a value of k. It will be shown later that under the assumption of fixed age-intervals, the number of units retired in each interval is a random variable which follows the multinomial distribution.[1]

The interval of time between t_k and t_{k+1} (i.e., h_k) is often defined in life studies of physical property as one year. This convention (which ignores retirement activity in age-interval $0 - \frac{1}{2}$) originated from the use of orthogonal polynomials in estimating the parameters of a hazard function. This method can best be applied if the age-intervals are equally spaced. There is nothing, however, in the definition of the probabilities expressed in a life table which fixes the width of these intervals. They may be chosen to suit the needs of the problem.[2]

It should also be noted that the last age-interval in which a sample of retirement data is grouped extends theoretically to infinity. Hence,

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[1] Infra, p. 42.

[2] It is noted by Reed and Merrell (56) that the term "complete life table" is used by actuaries to designate a table in which the interval is one year, and probabilities are stated for every year of age. This, however, is purely convention, since a table computed for monthly intervals would be more complete and one for weekly intervals still more so.

life table estimates that are a function of the width of an age-interval are undefined for the last interval.

The notation used to describe an age-interval can be extended to provide a mathematical description of the elements of a life table. The general format of a life table is given in Table 1. Entries in the life table are defined as follows:

(i) Mid-point (t_{mk}). The mid-point of the k^{th} age-interval such that $t_{mk} = (t_k + t_{k+1})/2$; $k = 1, 2, \ldots, n-1$, where n is the last age-interval in which retirement data are grouped.

(ii) Width (h_k). The width of the k^{th} age-interval such that $h_k = t_{k+1} - t_k$; $k = 1, 2, \ldots, n-1$. The width of the last interval, h_n, in theory, is infinite; no estimates of the hazard function or survivorship function can be obtained for this interval.

(iii) Number entering the k^{th} age-interval (N_k). The number of units entering the first age-interval is N_1, the total number of units placed in service as a group or vintage at age zero. In life studies of physical property it is assumed that all losses or withdrawals are actual retirements from service; so-called "right-censored" observations are not considered. Therefore, N_k is the number of units exposed to the risk of failure or retirement at the start of the k^{th} age-interval.

(iv) Number retired (d_k). This is the number of units retired during the k^{th} age-interval; thus, $d_k = N_k - N_{k+1}$; $k = 1, 2, \ldots, n-1$.

(v) Conditional proportion retired ($\hat{q}_k$). This is the estimated probability of retirement during the k^{th} age-interval, conditioned

Table 1. The Life Table

Age-Interval	Mid-point	Width	Number Entering Interval	Number Retired	Conditional Proportion Retired	Conditional Proportion Surviving	Cumulative Proportion Surviving	Estimated Probability Density Function	Estimated Hazard Function
t_1-t_2	t_{m1}	h_1	N_1	d_1	$\hat{q}_1$	$\hat{p}_1$	$\hat{S}_1=1.0$	$\hat{f}(t_{m1})$	$\hat{\lambda}_1$
t_2-t_3	t_{m2}	h_2	N_2	d_2	$\hat{q}_2$	$\hat{p}_2$	$\hat{S}_2$	$\hat{f}(t_{m2})$	$\hat{\lambda}_2$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
t_k-t_{k+1}	t_{mk}	h_k	N_k	d_k	$\hat{q}_k$	$\hat{p}_k$	$\hat{S}_k$	$\hat{f}(t_{mk})$	$\hat{\lambda}_k$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t_{n-1}-t_n$	t_{mn-1}	h_{n-1}	N_{n-1}	d_{n-1}	$\hat{q}_{n-1}$	$\hat{p}_{n-1}$	$\hat{S}_{n-1}$	$\hat{f}(t_{mn-1})$	$\hat{\lambda}_{n-1}$
$t_n-\infty$	--	--	N_n	d_n	1.0	0	$\hat{S}_n$	--	--

Where: $t_{mk} = (t_k + t_{k+1})/2; \quad k = 1, \ldots, n-1.$

$h_k = t_{k+1} - t_k; \quad k = 1, \ldots, n-1.$

$\hat{q}_k = d_k/N_k; \quad k = 1, \ldots, n-1.$

$\hat{p}_k = 1 - \hat{q}_k = N_{k+1}/N_k; \quad k = 1, \ldots, n-1.$

$\hat{S}_k = N_k/N_1 = \hat{p}_{k-1}\hat{S}_{k-1}; \quad k = 2, \ldots, n$

$= 1.0; \quad k = 1.$

$\hat{f}(t_{mk}) = (\hat{S}_k - \hat{S}_{k+1})/h_k; \quad k = 1, \ldots, n-1.$

$\hat{\lambda}_k = g(\hat{p}_k, \hat{q}_k); \quad k = 1, \ldots, n-1.$

upon exposure to the risk or forces of retirement at the start of the k^{th} interval. By definition,

$$\hat{q}_k = \frac{N_k - N_{k+1}}{N_k} = \frac{d_k}{N_k}; \qquad k = 1, 2, \ldots, n-1. \qquad (1)$$

In depreciation applications, $\hat{q}_k$ is commonly termed a "retirement ratio".

(vi) Conditional proportion surviving ($\hat{p}_k$). This is the estimated probability of surviving the k^{th} age-interval, conditioned upon exposure to the risk or forces of retirement at the start of the k^{th} interval. Thus, by definition,

$$\hat{p}_k = 1 - \hat{q}_k = \frac{N_{k+1}}{N_k}; \qquad k = 1, 2, \ldots, n-1. \qquad (2)$$

In depreciation applications, $\hat{p}_k$ is commonly termed a "survivor ratio".

(vii) Cumulative proportion surviving ($\hat{S}_k$). This is an estimate of the probability of surviving to the start of the k^{th} age-interval. The estimate is given by

$$\hat{S}_k = \hat{p}_{k-1}\,\hat{S}_{k-1} = \frac{N_k}{N_1}; \qquad k = 2, 3, \ldots, n$$

$$= 1.0 \qquad k = 1. \qquad (3)$$

This is a well-known life table estimate that is based on the fact that surviving to the start of the k^{th} age-interval means surviving to the start of the $(k-1)^{th}$ interval and then surviving the $(k-1)^{th}$ interval. This probability is defined for the last interval.

(iix) Estimated probability density function $\hat{f}(t_{mk})$. This is the estimated probability of retirement during the k^{th} age-interval per unit width. This estimate is given by

$$\hat{f}(t_{mk}) = \frac{\hat{S}_k - \hat{S}_{k+1}}{h_k}; \qquad k = 1, 2, \ldots, n-1. \tag{4}$$

Also, from the definition of $\hat{p}_k$ and $\hat{q}_k$ it follows that

$$\hat{f}(t_{mk}) = \frac{\hat{P}_k \hat{q}_k}{h_k}; \qquad k = 1, 2, \ldots, n-1. \tag{5}$$

(ix) Estimated hazard function $(\hat{\lambda}_k)$. This is an estimate of the hazard function for the k^{th} age-interval. In the literature of reliability theory, estimates of the hazard function are called hazard rates -- a term which will be adopted here and discussed further under the heading "Estimates of the Hazard Function". Generally, $\hat{\lambda}_k$ is a function of $\hat{p}_k$ and $\hat{q}_k$. Thus, $\hat{\lambda}_k$ will presently be expressed as

$$\hat{\lambda}_k = g(\hat{p}_k, \hat{q}_k); \qquad k = 1, 2, \ldots, n-1. \tag{6}$$

The Survival Functions

The functional relationship between the probability density function, the cumulative distribution function, the survivorship function, and the hazard function has been described by Broadbent (8), Cox (15), Gehan (25), and Jordan (38), among others. Collectively, these functions are known as "survival functions" or "biometric functions". To derive these functions, let T represent the life of a unit of property where T is measured from the installation date of the property to the date of its final retirement from service. We assume that T is a continuous random variable with one-dimensional sample space $S_t = \{t;\ 0 \leq t < \infty\}$. The survival functions are then defined as follows:

(i) Probability density function (p.d.f.), f(t). Since T is a continuous random variable, there exists a real-valued, non-negative function f(t), called the p.d.f., such that

(a) if K is the set $\{t;\ t_1 < t \leq t_2\}$, then the probability that T is in K, or the probability that a unit of property is retired between t_1 and t_2 is given by

$$\Pr[t_1 < T \leq t_2] = \int_{t_1}^{t_2} f(x)\,dx, \quad 0 \leq t_1 < t_2 < \infty$$

and

(b)

$$\Pr[0 < T < \infty] = \int_0^{\infty} f(x)\,dx = 1.0$$

where

$$f(t) = \lim_{\Delta t \to 0} \frac{\Pr[t < T \leq t + \Delta t]}{\Delta t}. \tag{7}$$

Thus, $f(t)$ is the instantaneous probability of retirement at age t.

(ii) Cumulative distribution function (c.d.f.), $F(t)$. The c.d.f., $F(t)$ is defined as the probability that a unit of property is retired before age t and is given by

$$F(t) = \Pr[T \leq t], \qquad t \geq 0.$$

Thus,

$$F(t) = \begin{cases} 0, & t \leq 0 \\ \displaystyle\int_0^t f(x)\,dx, & t > 0 \end{cases} . \tag{8}$$

Note that

$$f(t) = \frac{dF(t)}{dt} .$$

(iii) Survivorship function (s.f.), $S(t)$. The s.f., $S(t)$ is defined as the probability that a unit of property survives (i.e., remains in service) beyond age t and is given by

$$\begin{aligned} S(t) &= \Pr[T > t] \\ &= 1.0 - F(t). \end{aligned}$$

Thus,

$$S(t) = \begin{cases} 1.0, & t \leq 0 \\ \int_t^{\infty} f(x)\, dx, & t > 0 \end{cases} . \tag{9}$$

(iv) Hazard function (h.f.), $\lambda(t)$. The h.f., $\lambda(t)$ is the probability of nearly immediate retirement from service for a unit of property that is known to be in service at age t. That is,

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)} . \tag{10}$$

Now, from Equation 9 it is clear that

$$\frac{dS(t)}{dt} = -\frac{dF(t)}{dt}$$

and from Equation 8 that

$$-\frac{dF(t)}{dt} = -f(t).$$

These results can be combined with Equation 10 to obtain

$$\lambda(t) = \frac{f(t)}{S(t)} = -\frac{1}{S(t)}\frac{dS(t)}{dt}$$

Thus,

$$-\lambda(t) = \frac{d \ln S(t)}{dt}$$

and

$$\ln S(t) = -\int_0^t \lambda(x)dx$$

or

$$S(t) = e^{-\int_0^t \lambda(x)dx} \tag{11}$$

Let $\Delta(t)$ denote the cumulative hazard function, that is,

$$\Delta(t) = \int_0^t \lambda(x)dx.$$

Then

$$S(t) = \exp\{-\Delta(t)\}$$

and from Equations 8 - 10

$$F(t) = 1 - S(t) = 1 - \exp\{-\Delta(t)\} \tag{12}$$

and

$$\begin{aligned} f(t) = \frac{dF(t)}{dt} &= \exp\{-\Delta(t)\}\frac{d\Delta(t)}{dt} \\ &= \lambda(t)\exp\{-\Delta(t)\}. \end{aligned} \tag{13}$$

Thus, given any one of the four survival functions, the other three can be derived from equivalent functions. If $S(t)$ or $F(t)$ is given, $f(t)$ is obtained by differentiation and $\lambda(t)$ is obtained using Equation 10. If a form of $f(t)$ is given, then $F(t)$ is obtained using Equation 8, $S(t)$ is obtained using Equation 9, and $\lambda(t)$ is obtained from Equation 10. Similarly, if $\lambda(t)$ is given, $S(t)$ is obtained from Equation 11, $F(t)$ from Equation 12, and $f(t)$ from Equation 13.

ESTIMATES OF THE HAZARD FUNCTION

The purpose of this section is to discuss certain nonparametric methods for estimating the hazard function for each age-interval. In particular, we are seeking a sample estimate of the hazard rate for each age-interval that can be used to obtain estimates of the parameters of a hazard function (presently unspecified) using ordinary regression methods. We are also interested in the efficiency of the parameters estimated by a weighted least squares procedure where the weights w_k are either 1.0, $1/\widehat{Var}(\hat{\lambda}_k)$, or $N_k h_k$. It is necessary, therefore, to obtain an estimate of the variance of the hazard rate (i.e., $\widehat{Var}(\hat{\lambda}_k)$ for each of the methods used to estimate the hazard function.

Conditional Proportion Retired

When the underlying mathematical law of mortality is unknown, the survivorship function S(t), and hence the hazard function λ(t), can be estimated from the values $\hat{S}_k$ and $\hat{\lambda}_k$ respectively, in the life table. One of the simplest functions for $\hat{\lambda}_k$ is obtained by substituting the life table estimates for f(t) and S(t) into Equation 10, i.e., to use the estimate

$$\hat{\lambda}_k = \frac{\hat{f}(t_{mk})}{\hat{S}_k} .$$

It should be noted that this estimate is the ratio of the estimated p.d.f. at the mid-point t_{mk} of the k^{th} age-interval and the cumulative proportion surviving at t_k, the beginning of the k^{th} age-interval. Using Equation 4 we can write

$$\hat{\lambda}_k = \frac{\hat{S}_k - \hat{S}_{k+1}}{h_k \hat{S}_k}$$

and from Equation 3,

$$\hat{\lambda}_k = \frac{1}{h_k} \frac{N_k - N_{k+1}}{N_k} = \frac{1}{h_k} \frac{d_k}{N_k} .$$

But, from Equation 1, d_k/N_k is $\hat{q}_k$, the conditional proportion retired. Thus,

$$\hat{\lambda}_k = \frac{\hat{q}_k}{h_k} \; ; \quad k = 1, 2, \ldots, n-1. \tag{14}$$

The conditional proportion retired is commonly used by depreciation engineers as an estimate of the hazard function. However, it is seldom, if ever, used by researchers in other fields. This observation is supported by a rather extensive literature search in the field of actuarial statistics and the biomedical sciences in which no example could be found where the conditional proportion retired (dying) was used as an estimate of the hazard function. By the same token, no example could be found in the literature of depreciation where an estimate other than the conditional proportion retired was used.

An estimate of the variance of $\hat{\lambda}_k$ can be obtained from the sampling distribution of N_k (the number of units entering the k^{th} age-interval) and d_k (the number of units retired during the k^{th} age-interval). The number of units entering the first age-interval (i.e., N_1) can be viewed as N_1 independent trials of a random experiment where each trial can have one of several outcomes. The "outcome" of a particular unit (trial) may be retirement during the first age-interval, the second age-interval, . . ., or the n^{th} age-interval. Let $d_1, d_2, \ldots, d_n$ denote the number of units retired during the first age-interval, the second age-interval, . . ., and the n^{th} age-interval respectively. Also, let θ_k denote the probability that a unit is retired during the k^{th} age-interval ($k = 1, 2, \ldots, n$). Thus,

$$\theta_k = E[\hat{q}_k \hat{S}_k]$$

where E is expected value. Assuming that the N_1 units act independently, it can be shown that the n-dimensional random vector $(d_1, d_2, \ldots, d_n)$ is a multinomial random variable with parameters $(N_1; \theta_1, \theta_2, \ldots, \theta_n)$. Thus,

$$E[d_k] = N_1\theta_k; \qquad k = 1, 2, \ldots, n$$

$$Var(d_k) = N_1\theta_k(1 - \theta_k)$$

$$Cov(d_j, d_k) = -N_1\theta_j\theta_k; \quad j \neq k.$$

It can also be shown that N_k, the number of units surviving to the beginning of the k^{th} age-interval is a binomial random variable such that

$$E[N_k] = N_1 \sum_{i=k}^{n} \theta_i \equiv N_1(1 - \phi_k)$$

$$\mathrm{Var}(N_k) = N_1\left(\sum_{i=k}^{n} \theta_i\right)\left(\sum_{i=1}^{k-1} \theta_i\right)$$

$$= N_1(1 - \phi_k)(\phi_k)$$

where

$$\phi_k = \sum_{i=1}^{k-1} \theta_i \,.$$

Consider the random variable $\hat{q}_k = d_k/N_k$, which is the proportion of those units surviving to the start of the k^{th} age-interval that are retired during the k^{th} interval. Then,

$$E[\hat{q}_k] = E_{N_k}\left[E\left[\frac{d_k}{N_k} \mid N_k\right]\right]$$

and

$$\mathrm{Var}(\hat{q}_k) = E_{N_k}\left[\mathrm{Var}\left(\frac{d_k}{N_k} \mid N_k\right)\right] + \mathrm{Var}_{N_k}\left(E\left[\frac{d_k}{N_k} \mid N_k\right]\right)$$

where E_{N_k} is the expectation and Var_{N_k} is the variance with respect to the random variable N_k. Now, it can be shown that the conditional distribution of d_k, d_{k+1}, . . ., d_n given N_k is multinomial with parameters (N_k; q_k, q_{k+1}, . . ., q_{n-1}) where $q_k = \theta_k/(1 - \phi_k)$. Therefore,

$$E[d_k \mid N_k] = N_k q_k$$

$$\mathrm{Var}(d_k \mid N_k) = N_k q_k (1 - q_k)$$

$$\mathrm{Cov}(d_j, d_k) = -N_k q_j q_k; \quad j \neq k.$$

Hence,

$$E[\hat{q}_k] = E_{N_k}\left[E\left[\frac{d_k}{N_k} \mid N_k\right]\right] = E_{N_k}\left[\frac{1}{N_k} N_k q_k\right] = E_{N_k}[q_k] = q_k$$

and

$$\mathrm{Var}(\hat{q}_k) = E_{N_k}\left[\frac{1}{N_k^2} N_k q_k (1 - q_k)\right] + \cancel{\mathrm{Var}_{N_k}\left(\frac{1}{N_k} N_k q_k\right)} \rightarrow 0$$

$$= E_{N_k}\left[\frac{1}{N_k} q_k (1 - q_k)\right]$$

$$= q_k(1 - q_k) E\left[\frac{1}{N_k}\right].$$

But, using the Taylor series expansion which is applicable when N_1 is large,

$$E\left[\frac{1}{N_k}\right] = \frac{1}{E[N_k]} \doteq \frac{1}{N_1(1 - \phi_k)}\left[1 + \frac{\phi_k}{N_1(1 - \phi_k)}\right] \doteq \frac{1}{N_1(1 - \phi_k)} \qquad (15)$$

Thus, an approximate value of the variance of $\hat{q}_k$ is

$$\mathrm{Var}(\hat{q}_k) = \frac{q_k(1 - q_k)}{N_1(1 - \phi_k)} .$$

However, estimates of these parameters are

$$\hat{q}_k = \frac{d_k}{N_k},$$

and

$$\widehat{N_1(1 - \phi_k)} = N_k.$$

Therefore,

$$\widehat{\text{Var}}(\hat{q}_k) = \frac{\hat{q}_k(1 - \hat{q}_k)}{N_k} \qquad (16)$$

Having obtained an estimate of the variance of $\hat{q}_k$, an estimate of the variance of $\hat{\lambda}_k$ is given by

$$\widehat{\text{Var}}(\hat{\lambda}_k) = \widehat{\text{Var}}\left(\frac{\hat{q}_k}{h_k}\right) = \frac{1}{h_k^2}\,\widehat{\text{Var}}(\hat{q}_k).$$

Thus, for large samples,

$$\widehat{\text{Var}}(\hat{\lambda}_k) = \frac{\hat{q}_k(1 - \hat{q}_k)}{h_k^2 N_k} = \frac{\hat{q}_k \hat{p}_k}{h_k^2 N_k}. \qquad (17)$$

It should also be noted that the expected value of $\hat{\lambda}_k$ (i.e., $E[\hat{\lambda}_k]$) is given by

$$E[\hat{\lambda}_k] = E\left[\frac{\hat{q}_k}{h_k}\right] = \frac{1}{h_k} E[\hat{q}_k] = \frac{q_k}{h_k} \qquad (18)$$

which was implicitly used to obtain the variance of $\hat{q}_k$.

Actuarial Estimate

The so-called actuarial estimate of $\lambda(t)$ is considered by Gehan (25), Kimball (41), and Watson and Leadbetter (58), among others. This estimate can also be derived by substituting life table estimates of f(t) and S(t) into Equation 10 which defines the hazard function $\lambda(t)$. However, rather than estimating S(t) by $\hat{S}_k$, it is assumed that S(t) can be expressed as a linear function over the interval h_k such that plant retirements are distributed uniformly between t_k and t_{k+1}. It is reasonable, therefore, to estimate S(t) by the average cumulative proportion surviving at the mid-point of the interval. Thus, the actuarial estimate of $\lambda(t)$ is

$$\hat{\lambda}_k = \frac{\hat{f}(t_{mk})}{\hat{S}(t_{mk})}$$

$$= \frac{2\hat{f}(t_{mk})}{\hat{S}_k + \hat{S}_{k+1}} .$$

Using Equation 4, the actuarial estimate of $\lambda(t)$ can be written as

$$\hat{\lambda}_k = \frac{2(\hat{S}_k - \hat{S}_{k+1})}{h_k(\hat{S}_k + \hat{S}_{k+1})}$$

and, from Equation 3,

$$\hat{\lambda}_k = \frac{2(N_k - N_{k+1})/N_k}{h_k(N_k + N_{k+1})/N_k} = \frac{2d_k}{h_k(N_k + N_{k+1})}$$

But, from Equation 2, $(N_k - N_{k+1})/N_k$ is $1 - \hat{p}_k$ and $(N_k + N_{k+1})/N_k$ is $1 + \hat{p}_k$. Making these substitutions, we obtain

$$\hat{\lambda}_k = \frac{2(1 - \hat{p}_k)}{h_k(1 + \hat{p}_k)} = \frac{2\hat{q}_k}{h_k(1 + \hat{p}_k)} . \quad (19)$$

In words, the actuarial estimate is the number of retirements per unit time in the interval divided by the average number of survivors during the interval. This form is most often used in the biomedical sciences when the ages at death within the interval are not known.

The expected value of $\hat{\lambda}_k$ can be obtained from a restatement of Equation 19 in which d_k/N_k is substituted for $\hat{q}_k$ and N_{k+1}/N_k is substituted for $\hat{p}_k$. Thus,

$$\hat{\lambda}_k = \frac{d_k}{h_k(N_k - \frac{d_k}{2})} .$$

This estimate can be treated as a function, $f(d_k, N_k - \frac{d_k}{2})$, of two random variables and using a Taylor series expansion up to the second term (i.e., $i = 2$),

$$E[\hat{\lambda}_k] = \frac{\mu_{d_k}}{h_k\left[\mu_{(N_k - \frac{d_k}{2})}\right]} \left\{ 1 - \frac{\mathrm{Cov}\left[d_k, (N_k - \frac{d_k}{2})\right]}{\mu_{d_k}\left[\mu_{(N_k - \frac{d_k}{2})}\right]} + \frac{\mathrm{Var}\left[(N_k - \frac{d_k}{2})\right]}{\mu^2_{(N_k - \frac{d_k}{2})}} \right\}$$

where μ_{d_k} denotes the expected value of d_k and $\mu_{(N_k - \frac{d_k}{2})}$ denotes the expected value of $(N_k - \frac{d_k}{2})$. Using the conditioning argument that

$$E[\hat{\lambda}_k] = E_{N_k}[E(\hat{\lambda}_k \mid N_k)]$$

where E_{N_k} is the expectation with respect to the random variable N_k, and the fact that conditional on N_k, d_k is a binomial (N_k, q_k) random variable,

$$E[\hat{\lambda}_k \mid N_k] \doteq \frac{q_k}{h_k(1 - \frac{q_k}{2})}\left[1 + \frac{(1 - q_k)}{2N_k(1 - \frac{q_k}{2})} + \frac{q_k(1 - q_k)}{4(1 - \frac{q_k}{2})^2}\right]$$

and

$$E[\hat{\lambda}_k] \doteq \frac{q_k}{h_k(1 - \frac{q_k}{2})}\left\{1 + \frac{q_k(1 - q_k)}{4(1 - \frac{q_k}{2})^2} + \frac{(1 - q_k)}{2(1 - \frac{q_k}{2})} E\left[\frac{1}{N_k}\right]\right\}$$

where $q_k = \theta_k/(1 - \phi_k)$. Since N_k is a binomial $[N_1, (1 - \phi_k)]$ random variable,

$$E\left[\frac{1}{N_k}\right] \doteq \frac{1}{N_1(1 - \phi_k)}\left[1 + \frac{\phi_k}{N_1(1 - \phi_k)}\right] .$$

Therefore,

$$E[\hat{\lambda}_k] \doteq \frac{q_k}{h_k(1 - \frac{q_k}{2})}\left\{1 + \frac{q_k(1 - q_k)}{4(1 - \frac{q_k}{2})^2} + \frac{(1 - q_k)}{2(1 - \frac{q_k}{2})} \frac{1}{N_1(1 - \phi_k)} \left[1 + \frac{\phi_k}{N_1(1 - \phi_k)}\right]\right\} . \qquad (20)$$

An estimate of the variance of $\hat{\lambda}_k$ is suggested by Gehan (25) whose development proceeds as follows:

Let

$$\theta_k = E[\hat{q}_k \hat{S}_k], \qquad v_k = E[d_k],$$

$$\phi_k = \sum_{i=1}^{k-1} \theta_i, \qquad u_k = \sum_{i=1}^{k-1} v_i, \qquad m_k = \sum_{i=1}^{k-1} d_i,$$

$$\delta d_k = d_k - v_k, \qquad \delta m_k = m_k - u_k$$

where E is expected value. Now, if a sample of N_1 units is followed until all are retired and each retirement is recorded as occurring in one of n fixed intervals, the joint distribution of the number retired is multinomial. From Equation 19 the estimate of the hazard rate for the k^{th} age-interval can be written as

$$\hat{\lambda}_k = \frac{2\hat{q}_k}{h_k(1 + \hat{p}_k)} = \frac{d_k}{h_k(N_k - d_k/2)}$$

and

$$\mathrm{Var}\left[\frac{d_k}{h_k(N_1 - m_k - d_k/2)}\right] =$$

$$\mathrm{Var}\left\{\frac{v_k\left[1 - \frac{\delta d_k}{v_k}\right]}{h_k(N_1 - u_k - v_k/2)\left[1 - \frac{\delta m_k}{(N_1 - u_k - v_k/2)} - \frac{\delta d_k}{2(N_1 - u_k - v_k/2)}\right]}\right\}$$

and this is approximately

$$\approx \mathrm{Var}\left[\frac{v_k}{h_k(N_1 - u_k - v_k/2)}\left(1 + \frac{\delta d_k}{v_k} + \frac{\delta m_k}{(N_1 - u_k - v_k/2)} + \frac{\delta d_k}{2(N_1 - u_k - v_k/2)}\right)\right]$$

$$\approx \left[\frac{v_k}{h_k(N_1 - u_k - v_k/2)}\right]^2 E\left[\frac{(\delta d_k)^2}{v_k^2} + \frac{(\delta m_k)^2}{(N_1 - u_k - v_k/2)^2} + \frac{(\delta d_k)^2}{4(N_1 - u_k - v_k/2)^2}\right.$$

$$\left. + \frac{2(\delta d_k)(\delta m_k)}{v_k(N_1 - u_k - v_k/2)} + \frac{(\delta d_k)^2}{v_k(N_1 - u_k - v_k/2)} + \frac{(\delta m_k)(\delta d_k)}{(N_1 - u_k - v_k/2)^2}\right]$$

since $E[\delta d_k] = E[\delta m_k] = 0$.

With the assumption of multinomial sampling,

$$E[\delta d_k]^2 = N_1\theta_k(1 - \theta_k), \qquad E[\delta m_k]^2 = N_1\phi_k(1 - \phi_k)$$

and $E[\delta d_k \delta m_k] = -N_1\theta_k\phi_k$. Making these substitutions and after considerable simplification, we obtain

$$\mathrm{Var}(\hat{\lambda}_k) \approx \frac{\theta_k}{N_1 h_k^2(1 - \phi_k - \theta_k/2)^2}\left[1 - \left[\frac{\theta_k}{2(1 - \phi_k - \theta_k/2)^2}\right]^2\right] .$$

This formula assumes complete ascertainment of survival times. For incomplete samples we use

$$\theta_k \doteq \hat{S}_k\hat{q}_k \qquad \phi_k \doteq 1 - \hat{S}_k$$

where $\doteq$ means is estimated by. With these assumptions and replacements, the estimated variance of $\hat{\lambda}_k$ becomes

$$\mathrm{Var}(\hat{\lambda}_k) \simeq \frac{\hat{\lambda}_k^{\,2}}{N_1 \hat{q}_k} \left[1 - \left[\frac{h_k \hat{\lambda}_k}{2} \right]^2 \right] . \tag{21}$$

Maximum Likelihood Estimate

It was shown earlier that the number of units retired during the k^{th} age-interval (d_k) from the units installed at time zero (N_1) is a multinomially distributed random variable. The likelihood of the sample can be written as

$$Ps = \frac{N_1!}{\prod_{k=1}^{n} d_k!} q_1^{d_1} (p_1 q_2)^{d_2} (p_1 p_2 q_3)^{d_3} \cdots (p_1 \cdots p_{n-1})^{d_n}$$

$$= \frac{N_1!}{\prod_{k=1}^{n} d_k!} q_1^{d_1} p_1^{d_2} q_2^{d_2} p_1^{d_3} p_2^{d_3} q_3^{d_3} \cdots p_1^{d_n} \cdots p_{n-1}^{d_n}$$

$$= \frac{N_1!}{\prod_{k=1}^{n} d_k!} q_1^{d_1} p_1^{d_2+d_3+\ldots+d_n} q_2^{d_2} p_2^{d_3+d_4+\ldots+d_n} \cdots q_{n-1}^{d_{n-1}} p_{n-1}^{d_n}$$

$$= \frac{N_1!}{\prod_{k=1}^{n} d_k!} q_1^{d_1} p_1^{N_2} q_2^{d_2} p_2^{N_3} \cdots q_{n-1}^{d_{n-1}} p_{n-1}^{N_n}$$

$$= \frac{N_1!}{\prod_{k=1}^{n} d_k!} \prod_{k=1}^{n-1} p_k^{N_{k+1}} \prod_{k=1}^{n-1} q_k^{d_k} \tag{22}$$

where p_k and q_k are the true conditional probabilities of surviving and retiring in the k^{th} age-interval, i.e., q_k is the probability of retirement during the k^{th} age-interval conditioned on the unit surviving to the k^{th} age-interval. Similarly, p_k is the probability of surviving the k^{th} age-interval conditioned on the unit surviving to the k^{th} age-interval.

We now consider a formulation of the hazard function that was suggested by Sacher (57) and used by Gehan and Siddiqui (26) to analyze survival data for patients with plasma cell myeloma. Our motivation for considering this model will become apparent when the results are used with Equation 22 to obtain a maximum likelihood estimate of the hazard rate for each age-interval.

Suppose that a sample of survival times is grouped into age-intervals that are small enough so that it is reasonable to assume that the hazard function is constant within each age-interval.[1] In other words, we assume that

$$\lambda(t) = \lambda_k; \qquad t_k < t \leq t_{k+1},\ k = 0, 1, \ldots, n-1.$$

Under these conditions, p_k can be written as

$$p_k = \Pr[T > t_k + h_k \mid T > t_k] = \frac{\Pr[T > t_k + h_k]}{\Pr[T > t_k]}$$

---

[1]This is not unreasonable for industrial applications since h_k (as defined on p. 31) is typically small in relation to the expected service life of a property unit at age zero. Furthermore, plant additions and retirements are usually recorded on an annual basis and treated as a mid-year occurrence for life studies and depreciation accounting. Sample data of this type would not, therefore, represent an increasing or decreasing hazard rate within an age-interval.

where h_k, as defined earlier, is the width of the k^{th} age-interval. Since $\lambda(t)$ is now taken to be a step-function, we can replace the integral in Equation 11 with a summation operator and write

$$p_k = \frac{Pr[T > t_k + h_k]}{Pr[T > t_k]} = \frac{\exp\left[-\sum_{i=1}^{k} \lambda_i h_i\right]}{\exp\left[\sum_{i=1}^{k-1} \lambda_i h_i\right]}$$

$$= \exp -\{\lambda_k h_k\}. \tag{23}$$

Similarly, using Equation 2 we can write

$$q_k = \frac{Pr[t_k < T \leq t_k + h_k]}{Pt[T > t_k]} = 1 - p_k$$

$$= 1 - \exp\{-\lambda_k h_k\} . \tag{24}$$

We now have a specification for p_k and q_k in terms of the hazard function which can be used with Equation 22 to obtain a maximum likelihood estimate of the hazard rate for each age-interval. Thus, making these substitutions for p_k and q_k in Equation 22, the likelihood of the sample becomes

$$Ps = \frac{N_1!}{\prod_{k-1}^{n} d_k} \prod_{k=1}^{n-1} e^{-\lambda_k h_k N_{k+1}} \prod_{k=1}^{n-1} (1 - e^{-\lambda_k h_k})^{d_k}$$

and taking the logarithm we obtain

$$L = \ln Ps = \ln N_1! - \sum_{k=1}^{n-1} \lambda_k h_k N_{k+1} + \sum_{k=1}^{n-1} d_k \ln(1 - e^{-\lambda_k h_k})$$

$$- \ln \prod_{k=1}^{n} d_k$$

The value of λ_k (i.e., $\hat{\lambda}_k$) which maximizes L can be found by differentiating with respect to λ_k and setting the derivative equal to zero,

$$\frac{\partial L}{\partial \lambda_k} = -h_k N_{k+1} + \frac{d_k h_k e^{-\lambda_k h_k}}{1 - e^{-\lambda_k h_k}} = 0; \qquad k = 1, 2, \ldots, n-1.$$

Thus, to solve for $\hat{\lambda}_k$ we must solve

$$-h_k N_{k+1} + \frac{d_k h_k e^{-\hat{\lambda}_k h_k}}{1 - e^{-\hat{\lambda}_k h_k}} = 0$$

from which we obtain

$$e^{-\hat{\lambda}_k h_k} = \frac{N_{k+1}}{N_{k+1} + d_k} .$$

Now, by definition, $N_{k+1} + d_k = N_k$ and, from Equation 2, $N_{k+1}/N_k = \hat{p}_k$. Therefore,

$$e^{-\hat{\lambda}_k h_k} = \hat{p}_k$$

and the maximum likelihood estimator for λ_k is

$$\hat{\lambda}_k = -\frac{1}{h_k} \ln \hat{p}_i. \tag{25}$$

Although Equation 25 was derived from a cohort life table which describes the retirement experience of a single vintage, it is a simple matter to extend this result to a series of cohorts in which the retirement experience of several vintages is combined to obtain an estimator for λ_k. To show this, we extend our notation to include double subscripts where the first subscript denotes a vintage and the second subscript denotes an age-interval measured from the installation date of the same vintage. Thus, $N_{j,k}$ for example, identifies the number of units from the j^{th} vintage entering the k^{th} age-interval.

Now, suppose that we have a homogeneous population in which each vintage is subject to the same forces of retirement and in which the conditional probability of retirement for one unit of property is not influenced by the retirement of any other unit in the group. Under these conditions N_1 can be viewed as the sum of all units entering the zero age-interval from all vintages included in the group. In other words,

$$N_k = \sum_{j=1}^{m} N_{j,k}. \tag{26}$$

It follows then, from our assumption of independence that Equation 22 can be viewed as the likelihood of a sample obtained from a random experiment that is repeated m times. We can, therefore, restate Equation 25 in terms of m multiple vintages without changing the likelihood function. Thus, using Equation 26, we obtain

$$\hat{\lambda}_k = -\frac{1}{h_k} \ln \hat{p}_k$$

$$= \frac{-\ln(N_{k+1}/N_k)}{h_k}$$

$$= \frac{-\ln(\sum_{j=1}^{m} N_{j,k+1} / \sum_{j=1}^{m} N_{j,k})}{h_k} \qquad (27)$$

The expected value of $\hat{\lambda}_k$ can be obtained from a restatement of Equation 24 in which $1 - d_k/N_k$ is substituted for $\hat{p}_k$. Thus,

$$\hat{\lambda}_k = -\frac{1}{h_k} \ln(1 - \frac{d_k}{N_k}).$$

Since $\hat{\lambda}_k = f(\hat{p}_k)$ is a function of the random variable $\hat{p}_k$, a Taylor series expansion about the expected value $\mu_{\hat{p}_k}$ of $\hat{p}_k$ yields

$$\hat{\lambda}_k = f(\mu_{\hat{p}_k}) + \sum_{i=1}^{\infty} \frac{1}{i} \frac{d^i f}{d\hat{p}_k^i} \Bigg|_{\mu_{\hat{p}_k}} (\hat{p}_k - \mu_{\hat{p}_k})^i$$

and

$$E[\hat{\lambda}_k] = f(\mu_{\hat{p}_k}) + \sum_{i=1}^{\infty} \frac{1}{i} \frac{d^i f}{d\hat{p}_k^i} \Bigg|_{\mu_{\hat{p}_k}} E[\hat{p}_k - \mu_{\hat{p}_k}].$$

If the Taylor series expansion is limited to two terms (i.e., i = 2), then

$$E[\hat{\lambda}_k] \doteq -\frac{1}{h_k}\ln(1 - q_k) + \frac{1}{2h_k(1 - q_k)^2} E[(\hat{p}_k - \mu_{\hat{p}_k})^2]$$

$$\doteq -\frac{1}{h_k}\ln(1 - q_k) + \frac{1}{2h_k(1 - q_k)^2} \mathrm{Var}(\hat{p}_k)$$

where

$$q_k = \theta_k/(1 - \phi_k) = \theta_k/\sum_{i=k}^{n} \theta_i .$$

But,

$$\mathrm{Var}(\hat{p}_k) = \mathrm{Var}(1 - \frac{d_k}{N_k}) = \mathrm{Var}(\frac{d_k}{N_k})$$

and

$$\mathrm{Var}(\frac{d_k}{N_k}) = E_{N_k}\left[\mathrm{Var}(\frac{d_k}{N_k} \mid N_k)\right] + \mathrm{Var}_{N_k}(E\left[\frac{d_k}{N_k} \mid N_k\right]).$$

Since $E\left[\frac{d_k}{N_k} \mid N_k\right] = q_k$, a constant, and

$$\mathrm{Var}(\frac{d_k}{N_k} \mid N_k) = \frac{q_k(1 - q_k)}{N_k},$$

it follows that

$$\mathrm{Var}(\frac{d_k}{N_k}) = q_k(1 - q_k)E\left[\frac{1}{N_k}\right].$$

From Equation 15,

$$E\left[\frac{1}{N_k}\right] \doteq \frac{1}{N_1(1-\phi_k)}$$

for N_1 large. Therefore,

$$\mathrm{Var}\left(\frac{d_k}{N_k}\right) = \frac{q_k(1-q_k)}{N_1(1-\phi_k)}$$

and

$$E[\hat{\lambda}_k] \doteq -\frac{1}{h_k}\ln(1-q_k) + \frac{q_k}{2h_k N_1(1-\phi_k)(1-q_k)} \tag{28}$$

An approximation of the variance of the maximum likelihood estimate can be obtained by considering the first term of a Taylor series expansion. Thus,

$$\mathrm{Var}(\hat{\lambda}_k) \doteq E[\hat{\lambda}_k - f(\mu_{\hat{p}_k})]^2 = \left(\frac{df}{d\hat{p}_k}\bigg|_{\mu_{\hat{p}_k}}\right)^2 \mathrm{Var}(\hat{p}_k)$$

$$\doteq \frac{1}{h_k^2(1-q_k)^2}\left[\frac{q_k(1-q_k)}{N_1(1-\phi_k)}\right]$$

$$\doteq \frac{q_k}{h_k^2 N_1(1-\phi_k)(1-q_k)} \; . \tag{29}$$

An estimate of the variance of $\hat{\lambda}_k$ (i.e., $\widehat{\mathrm{Var}}(\hat{\lambda}_k)$) can be obtained from the sample estimates of q_k and $N_1(1-\phi_k)$. These estimates are

$$\hat{q}_k = \hat{q}_k$$

and

$$\widehat{N_1(1 - \phi_k)} = N_k.$$

Thus,

$$\widehat{\mathrm{Var}}(\hat{\lambda}_k) = \frac{\hat{q}_k}{h_k^2 \; N_k \hat{p}_k} \tag{30}$$

provides an estimate of the variance of $\hat{\lambda}_k$ that can be used to obtain a weighted regression estimate of the parameters of a hazard function.

METHOD OF ANALYSIS

Having derived various estimates of the hazard rate for each age-interval, it would be helpful to know which, if any, of these estimates is in some sense "best" for depreciation applications. Since the variance of the estimated hazard rate is different for each age-interval, a related question becomes which, if any, method of weighting combined with a given estimator provides the best estimate of the parameters of an assumed hazard function.

To make this comparison, a Monte Carlo study was undertaken in which random samples were drawn from each of four different models of the hazard function $\lambda(t)$. The models chosen for this analysis include:

(i) $\lambda(t) = \lambda_0;$ $\quad \lambda_0 > 0$ (exponential distribution)

(ii) $\lambda(t) = \lambda_0 + \lambda_1 t;$ $\quad \lambda(t) > 0$ (linear hazard function)

(iii) $\lambda(t) = \exp\{\lambda_0 + \lambda_1 t\};$ $\quad \lambda(t) > 0$ (Gompertz distribution)

(iv) $\lambda(t) = \lambda_0 \lambda_1 t^{\lambda_1 - 1};$ $\quad \lambda_0, \lambda_1 > 0$ (Weibull distribution).

For each of these models, either the hazard function or its logarithmic transform is a function of the parameters λ_0, λ_1 and t (or ln t). Consequently, the parameters of these models can be estimated by least squares, or by weighted least squares since the variance of the estimated hazard rate is different for each age-interval.

The equations fitted to the samples drawn from the four models are:

(i) $\hat{\lambda}_k = \lambda_0$ (exponential distribution)

(ii) $\hat{\lambda}_k = \lambda_0 + \lambda_1 t_{mk}$ (linear hazard function)

(iii) $\ln \hat{\lambda}_k = \lambda_0 + \lambda_1 t_{mk}$ (Gompertz distribution)

(iv) $\ln \hat{\lambda}_k = \ln(\lambda_0 \lambda_1) + (\lambda_1 - 1) \ln t_{mk}$ (Weibull distribution) where for all models, $k = 1, 2, \ldots, n-1$.

The regression equations for all models can be written in the form

$$Y = T\lambda + \varepsilon$$

where Y is an $(n-1) \times 1$ vector of observed hazard rates (or their natural logs) taken from the life table; T is an $(n-1) \times j$ matrix $(j = 1, 2)$ which, depending on the model, contains ones and age-interval mid-points; λ is a $j \times 1$ vector of parameters; and ε is an $(n-1) \times 1$ vector of errors with expectation zero and sample variance matrix,

$$\hat{V} = \begin{bmatrix} \hat{v}(t_{m1}) & & 0 \\ & \ddots & \\ 0 & & \hat{v}(t_{m,n-1}) \end{bmatrix} .$$

This matrix is taken as diagonal since, as discussed earlier, it is not difficult to show that for large samples the covariances of the hazard rates are asymptotically zero. For the purpose of this study, the elements of $\hat{V}$ are estimated by $\widehat{Var}(\hat{\lambda}_k)$ which are given by Equation 17 when the elements of Y are estimated by Equation 14 (the conditional proportion retired); Equation 21 when the elements of Y are estimated by Equation 19 (the actuarial estimate); and Equation 30 when the elements of Y are estimated by Equation 25 (the maximum likelihood estimate). When the elements of Y are $\ln \hat{\lambda}_k$, $\hat{v}(t_{mk})$ is given by $\widehat{Var}(\hat{\lambda}_k)/\hat{\lambda}_k^2$.

A weighted least squares estimate of the elements of λ (i.e., the parameters of the underlying hazard function) can be obtained by minimizing

$$Z = (Y - T\lambda)'W(Y - T\lambda)$$

where

$$W = \begin{bmatrix} w_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & w_{n-1} \end{bmatrix}$$

is an (n-1) × (n-1) matrix of weights. The weights considered in this study are: 1.0, $1/\widehat{\text{Var}}(\hat{\lambda}_k)$, and $N_k h_k$.

It is well-known that the vector of least squares estimates of the parameters is given by

$$\hat{\lambda} = (T'WT)^{-1}T'WY$$

and the estimated variance-covariance matrix of $\hat{\lambda}$ by

$$V_{\hat{\lambda}} = L'\hat{V}L$$

where

$$L' = (T'WT)^{-1}T'W .$$

These calculations have been computerized by Kennedy (in Ref. 40) whose program was obtained from the Texas Medical Center and modified for the purpose of this study. A listing of the modified version of this program is contained in Appendix B. The general method of estimation of parameters can be described as follows: first, the program obtains sample estimates of the hazard rates using the conditional proportion retired, the actuarial estimate, or the maximum likelihood estimate for each age-interval over the observation period. From the sample estimates of the hazard rate for each age-interval, the program obtains estimates of the parameters for the four models (both weighted and unweighted) by ordinary regression methods. Finally, using the least squares estimates of the parameters, the program computes the hazard, survivorship, and probability density functions for each of the four models. Because the width of the last age-interval is theoretically infinite, estimates of the hazard and probability density functions are not defined in that interval. An additional feature of the program that was not incorporated in this study is the calculation of a χ^2 statistic that can be used in selecting the best fitting model.

A second computer program was used to draw random samples from the four hazard functions chosen for this analysis. The program was originally written by this author (62) to simulate the retirement experience of industrial property drawn from a population described by the Iowa-Type survivorship functions. The program was modified to accommodate the four hazard functions used in this analysis and linked via disk output to the "Actuarial" program for estimating parameters by the above regression methods. The technique used to generate aged retirements is the well-known Monte Carlo simulation procedure. A retirement is simulated by drawing a random number between 0.0 and 1.0 from a uniform distribution, where each number drawn represents a unit of property. The age of a retirement is determined by calculating the value of t associated with a specified cumulative distribution function that has an ordinate value equal in magnitude to the value of the random number. This process is repeated

N_1 times (i.e., the number of units installed at age zero) and a tally is kept of the number of units retired in each age-interval.

The population parameters assigned to the distributions (i.e., models) used in this study were selected to produce an average service life of approximately five years. This selection was viewed as a reasonable compromise between obtaining a sufficient number of age-intervals to conduct a meaningful analysis and minimizing the amount of computer time needed to generate a series of random samples and estimate the parameters. The values of the population parameters used in this study are as follows:

Model	λ_0	λ_1
(i) Exponential distribution	0.20	--
(ii) Linear hazard function	0.10	0.02
(iii) Gompertz distribution	-2.00	0.07
(iv) Weibull distribution	0.08	1.50 .

A secondary consideration in this study was whether or not a given estimator combined with a given method of weighting consistently provides a "best" estimate of the population parameters under varying degrees of censoring. This question was investigated by truncating a complete life table for each model at two levels of censoring and estimating parameters from the censored data. The two levels of censoring were arbitrarily selected to produce a "lightly censored" life table ending at about 20% surviving and a "heavily censored" life table ending at about 60% surviving. The value of the survivorship function containing the population parameters and the corresponding age at which the life table was truncated for each model is as follows:

Model	lightly censored age	S(t)	heavily censored age	S(t)
(i) Exponential distribution	8.5	18.27%	2.5	60.65%
(ii) Linear hazard function	8.5	20.75	3.5	62.34
(iii) Gompertz distribution	8.5	20.76	3.5	58.46
(iv) Weibull distribution	7.5	19.33	3.5	59.22 .

The results of this analysis are summarized in Tables 2 thru 13. Each of the 12 tables contains various estimates of the parameters and related statistics derived from 27 different analyses of a given model and degree of censoring. The first 3 tables provide a comparison of the average estimates obtained when the underlying distribution was exponential. Tables 5 thru 7 contain the averages of parameters estimated when the underlying distribution was a linear hazard function. The average estimates obtained when the underlying distribution was Gompertz is shown in Tables 8 thru 10, and the averages obtained from a Weibull distribution are shown in Tables 11 thru 13.

In total, 48,600 life tables were generated by drawing random samples containing either 100 or 1000 units installed at age zero. Parameters were estimated for both 100 and 1000 unit vintages in order to determine whether or not a given estimate of the hazard rate is sensitive to the size of the sample. An example of a generated life table, estimates of the parameters, and estimates of the hazard, survivorship, and probability density function is contained in Appendix C.

Each of the 12 tables is also partitioned according to the number of replications included in each study. Averages of the parameter estimates were computed from either 50 or 100 vintages (i.e., replications)

Table 2. Exponential distribution -- complete data

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.1822	.1464	.1758	.1825	.1452	.1763	.1844	.1768	.1819
Bias	-.0178	-.0536	-.0242	-.0175	-.0548	-.0237	-.0156	-.0232	-.0181
S.D. $\hat{\lambda}_0$	.0226	.0230	.0169	.0217	.0241	.0161	.0122	.0070	.0048
M.S.E.	.0286	.0582	.0294	.0278	.0598	.0286	.0197	.0242	.0187
Actuarial Estimate									
$\bar{\lambda}_0$	.2087	.1589	.1963	.2093	.1593	.1969	.2085	.1933	.1997
Bias	.0087	-.0411	-.0037	.0093	-.0407	-.0031	.0085	-.0067	-.0003
S.D. $\hat{\lambda}_0$	.0290	.0232	.0206	.0280	.0225	.0197	.0157	.0071	.0057
M.S.E.	.0300	.0471	.0207	.0294	.0465	.0198	.0177	.0097	.0057
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	.2111	.1594	.1975	.2117	.1600	.1982	.2104	.1938	.2004
Bias	.0111	-.0406	-.0025	.0117	-.0400	-.0018	.0104	-.0062	.0004
S.D. $\hat{\lambda}_0$	.0299	.0232	.0209	.0291	.0222	.0200	.0163	.0070	.0057
M.S.E.	.0316	.0466	.0208	.0312	.0457	.0200	.0192	.0093	.0057

Table 3. Exponential distribution -- lightly censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.1781	.1660	.1808	.1791	.1663	.1815	.1819	.1809	.1823
Bias	-.0219	-.0340	-.0192	.0209	-.0337	-.0185	-.0181	-.0191	-.0177
S.D.$\hat{\lambda}_0$	.0218	.0206	.0185	.0195	.0197	.0187	.0056	.0058	.0058
M.S.E.	.0307	.0396	.0265	.0285	.0390	.0262	.0189	.0199	.0186
Actuarial Estimate									
$\bar{\lambda}_0$	.1971	.1749	.2000	.1981	.1753	.2006	.1991	.1974	.1996
Bias	-.0029	-.0251	.0000	-.0019	-.0247	.0006	-.0009	-.0026	-.0004
S.D.$\hat{\lambda}_0$	.0267	.0245	.0225	.0240	.0228	.0229	.0067	.0069	.0069
M.S.E.	.0266	.0349	.0223	.0240	.0335	.0228	.0067	.0073	.0068
Maximum likelihood Estimate									
$\bar{\lambda}_0$	.1980	.1748	.2009	.1990	.1752	.2015	.1997	.1979	.2003
Bias	-.0020	-.0252	.0009	-.0010	-.0248	.0015	-.0003	-.0021	.0003
S.D.$\hat{\lambda}_0$	.0271	.0246	.0229	.0244	.0229	.0233	.0068	.0070	.0070
M.S.E.	.0269	.0350	.0227	.0243	.0337	.0232	.0067	.0072	.0069

Table 4. Exponential distribution -- heavily censored

	R=50 N$_1$=100			R=100 N$_1$= 100			R=50 N$_1$=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.1812	.1760	.1835	.1839	.1778	.1848	.1846	.1826	.1836
Bias	-.0188	-.0240	-.0165	-.0161	-.0222	-.0152	-.0154	-.0174	-.0164
S.D.$\hat{\lambda}_0$	.0253	.0259	.0246	.0250	.0256	.0248	.0086	.0086	.0084
M.S.E.	.0313	.0351	.0294	.0296	.0338	.0290	.0176	.0194	.0184
Actuarial Estimate									
$\bar{\lambda}_0$	.1976	.1882	.2012	.2004	.1906	.2026	.2000	.1986	.1999
Bias	-.0024	-.0118	.0012	.0004	-.0094	.0026	.0000	-.0014	-.0001
S.D.$\hat{\lambda}_0$	.0296	.0308	.0295	.0294	.0300	.0300	.0100	.0102	.0100
M.S.E.	.0294	.0327	.0292	.0293	.0313	.0300	.0099	.0102	.0099
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	.1982	.1883	.2020	.2011	.1908	.2033	.2005	.1991	.2004
Bias	-.0018	-.0117	.0020	.0011	-.0092	.0033	.0005	-.0009	.0004
S.D.$\hat{\lambda}_0$	.0298	.0310	.0299	.0297	.0301	.0303	.0101	.0102	.0101
M.S.E.	.0296	.0328	.0297	.0296	.0313	.0303	.0100	.0101	.0100

Table 5. Linear hazard function -- complete data

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.1069	.1016	.1078	.1083	.1034	.1082	.1142	.1022	.1012
Bias	.0069	.0016	.0078	.0083	.0034	.0082	.0142	.0022	.0012
S.D. $\hat{\lambda}_0$	.0343	.0254	.0212	.0389	.0265	.0248	.0265	.0088	.0088
M.S.E.	.0346	.0252	.0224	.0396	.0266	.0260	.0298	.0090	.0088
$\bar{\lambda}_1$	.0139	.0101	.0131	.0138	.0100	.0131	.0134	.0145	.0154
Bias	-.0061	-.0099	-.0069	-.0062	-.0100	-.0069	-.0066	-.0055	-.0046
S.D. $\hat{\lambda}_1$	.0072	.0061	.0051	.0080	.0064	.0056	.0042	.0020	.0018
M.S.E.	.0094	.0116	.0085	.0101	.0119	.0089	.0078	.0058	.0049
Actuarial Estimate									
$\bar{\lambda}_0$	.0984	.1008	.1075	.0999	.1024	.1080	.1016	.1028	.1013
Bias	-.0016	.0008	.0075	-.0001	.0024	.0080	.0016	.0028	.0013
S.D. $\hat{\lambda}_0$	.0471	.0272	.0248	.0536	.0270	.0300	.0406	.0094	.0108
M.S.E.	.0467	.0269	.0257	.0533	.0270	.0309	.0402	.0097	.0108
$\bar{\lambda}_1$	.0203	.0123	.0178	.0203	.0123	.0178	.0196	.0181	.0195
Bias	.0003	-.0077	-.0022	.0003	-.0077	-.0022	-.0004	-.0019	-.0005
S.D. $\hat{\lambda}_1$	.0106	.0066	.0065	.0116	.0063	.0072	.0066	.0021	.0023
M.S.E.	.0105	.0101	.0068	.0115	.0099	.0075	.0065	.0028	.0023
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	.0939	.1006	.1058	.0950	.1021	.1062	.0939	.1025	.1004
Bias	-.0061	.0006	.0058	-.0050	.0021	.0062	.0061	.0025	.0004
S.D. $\hat{\lambda}_0$	.0503	.0272	.0251	.0583	.0266	.0308	.0469	.0094	.0111
M.S.E.	.0502	.0269	.0255	.0582	.0265	.0313	.0468	.0096	.0110
$\bar{\lambda}_1$	.0216	.0124	.0186	.0217	.0124	.0186	.0212	.0183	.0200
Bias	.0016	-.0076	-.0014	.0017	-.0076	-.0014	.0012	-.0017	.0000
S.D. $\hat{\lambda}_1$	.0116	.0066	.0067	.0130	.0062	.0076	.0076	.0021	.0024
M.S.E.	.0116	.0100	.0068	.0130	.0098	.0077	.0076	.0027	.0024

Table 6. Linear hazard function -- lightly censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.0986	.0913	.0984	.0991	.0919	.0987	.0980	.0961	.0972
Bias	-.0014	-.0087	-.0016	-.0009	-.0081	-.0013	-.0020	-.0039	-.0028
S.D.$\hat{\lambda}_0$	.0299	.0299	.0261	.0287	.0267	.0257	.0096	.0085	.0088
M.S.E.	.0296	.0309	.0259	.0286	.0278	.0256	.0097	.0093	.0092
$\bar{\lambda}_1$	.0158	.0147	.0162	.0158	.0147	.0162	.0164	.0167	.0167
Bias	-.0042	-.0053	-.0038	-.0042	-.0053	-.0038	-.0036	-.0033	-.0033
S.D.$\hat{\lambda}_1$	.0093	.0092	.0080	.0084	.0080	.0073	.0027	.0022	.0023
M.S.E.	.0101	.0105	.0088	.0094	.0096	.0082	.0045	.0040	.0040
Actuarial Estimate									
$\bar{\lambda}_0$	.1007	.0941	.1010	.1014	.0947	.1014	.1004	.0987	.0998
Bias	.0007	-.0059	.0010	.0014	-.0053	.0014	.0004	-.0013	-.0002
S.D.$\hat{\lambda}_0$	.0355	.0334	.0305	.0342	.0295	.0303	.0112	.0094	.0102
M.S.E.	.0352	.0336	.0302	.0341	.0298	.0302	.0111	.0094	.0101
$\bar{\lambda}_1$	.0196	.0162	.0199	.0195	.0162	.0199	.0197	.0198	.0200
Bias	-.0004	-.0038	.0001	-.0005	-.0038	-.0001	-.0003	-.0002	.0000
S.D.$\hat{\lambda}_1$	.0119	.0109	.0100	.0106	.0096	.0091	.0033	.0026	.0028
M.S.E.	.0118	.0114	.0099	.0106	.0103	.0091	.0033	.0026	.0028
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	.1003	.0942	.1008	.1012	.0948	.1012	.1003	.0986	.0997
Bias	.0003	-.0058	.0008	.0012	-.0052	.0012	.0003	-.0014	-.0003
S.D.$\hat{\lambda}_0$	.0360	.0335	.0308	.0347	.0296	.0307	.0114	.0094	.0102
M.S.E.	.0356	.0337	.0305	.0345	.0299	.0306	.0113	.0094	.0101
$\bar{\lambda}_1$	.0199	.0161	.0202	.0197	.0161	.0202	.0199	.0199	.0201
Bias	-.0001	-.0039	.0002	-.0003	-.0039	.0002	-.0001	-.0001	.0001
S.D.$\hat{\lambda}_1$	.0122	.0109	.0102	.0109	.0096	.0093	.0033	.0026	.0028
M.S.E.	.0121	.0115	.0101	.0108	.0103	.0093	.0033	.0026	.0028

Table 7. Linear hazard function -- heavily censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	.0967	.0889	.0962	.0974	.0894	.0971	.0952	.0939	.0945
Bias	-.0003	-.0111	-.0038	-.0026	-.0106	-.0029	-.0048	-.0061	-.0055
S.D. $\hat{\lambda}_0$	.0394	.0384	.0372	.0355	.0363	.0348	.0125	.0124	.0124
M.S.E.	.0391	.0396	.0370	.0354	.0376	.0347	.0133	.0137	.0135
$\bar{\lambda}_1$	.0169	.0183	.0175	.0169	.0187	.0174	.0179	.0182	.0182
Bias	-.0031	-.0017	-.0025	-.0031	-.0013	-.0026	-.0021	-.0018	-.0018
S.D. $\hat{\lambda}_1$	.0218	.0216	.0209	.0211	.0218	.0207	.0071	.0069	.0069
M.S.E.	.0218	.0215	.0208	.0212	.0217	.0208	.0073	.0071	.0071
Actuarial Estimate									
$\bar{\lambda}_0$	.0999	.0912	.0997	.1006	.0914	.1006	.0977	.0967	.0972
Bias	-.0001	-.0088	-.0003	.0006	-.0086	.0006	-.0023	-.0033	-.0028
S.D. $\hat{\lambda}_0$	.0429	.0417	.0411	.0387	.0393	.0386	.0136	.0135	.0137
M.S.E.	.0425	.0422	.0407	.0385	.0400	.0384	.0137	.0138	.0138
$\bar{\lambda}_1$	.0203	.0210	.0208	.0203	.0216	.0207	.0212	.0213	.0214
Bias	.0003	.0010	.0008	.0003	.0016	.0007	.0012	.0013	.0014
S.D. $\hat{\lambda}_1$	.0244	.0241	.0236	.0238	.0245	.0236	.0079	.0078	.0078
M.S.E.	.0242	.0239	.0234	.0237	.0244	.0235	.0079	.0078	.0078
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	.1000	.0912	.0997	.1006	.0914	.1006	.0977	.0967	.0972
Bias	.0000	-.0088	-.0003	.0006	-.0086	.0006	-.0023	-.0033	-.0028
S.D. $\hat{\lambda}_0$	.0430	.0418	.0412	.0388	.0394	.0387	.0136	.0135	.0137
M.S.E.	.0426	.0423	.0408	.0386	.0401	.0385	.0137	.0138	.0138
$\bar{\lambda}_1$	.0204	.0210	.0209	.0205	.0217	.0209	.0214	.0214	.0216
Bias	.0004	.0010	.0009	.0005	.0017	.0009	.0014	.0014	.0016
S.D. $\hat{\lambda}_1$	.0245	.0242	.0237	.0239	.0246	.0237	.0080	.0078	.0078
M.S.E.	.0243	.0240	.0235	.0238	.0245	.0236	.0080	.0078	.0079

Table 8. Gompertz hazard function -- complete data

	R=50 N_1=100			R=100 N_1=100			R=5- N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	-2.0919	-1.9953	-2.0717	-2.0694	-1.9885	-2.0571	-1.9957	-2.0505	-2.0462
Bias	-0.0919	0.0047	-0.0717	-0.0694	0.0115	-0.0571	0.0043	-0.0505	-0.0462
S.D. $\hat{\lambda}_0$	0.1820	0.1414	0.1584	0.1773	0.1491	0.1566	0.0806	0.0497	0.0530
M.S.E.	0.2023	0.1401	0.1724	0.1896	0.1488	0.1659	0.0799	0.0705	0.0699
$\bar{\lambda}_1$	0.0511	0.0616	0.0470	0.0498	0.0608	0.0464	0.0502	0.0631	0.0587
Bias	-0.0189	-0.0084	-0.0230	-0.0202	-0.0092	-0.0236	-0.0198	-0.0069	-0.0113
S.D. $\hat{\lambda}_1$	0.0267	0.0224	0.0262	0.0272	0.0259	0.0266	0.0115	0.0079	0.0087
M.S.E.	0.0325	0.0237	0.0347	0.0338	0.0274	0.0355	0.0228	0.0104	0.0142
Actuarial Estimate									
$\bar{\lambda}_0$	-2.0616	-1.9612	-2.0237	-2.0383	-1.9544	-2.0095	-1.9570	-1.9974	-1.9923
Bias	-0.0616	0.0388	-0.0237	-0.0383	0.0456	-0.0095	0.0430	0.0026	0.0077
S.D. $\hat{\lambda}_0$	0.1925	0.1466	0.1658	0.1862	0.1517	0.1642	0.0882	0.0543	0.0570
M.S.E.	0.2003	0.1502	0.1658	0.1892	0.1577	0.1637	0.0973	0.0538	0.0570
$\bar{\lambda}_1$	0.0633	0.0724	0.0566	0.0619	0.0719	0.0563	0.0601	0.0708	0.0665
Bias	-0.0067	0.0024	-0.0134	-0.0081	0.0019	-0.0137	-0.0099	0.0008	-0.0035
S.D. $\hat{\lambda}_1$	0.0285	0.0223	0.0273	0.0292	0.0258	0.0279	0.0129	0.0089	0.0096
M.S.E.	0.0290	0.0222	0.0302	0.0302	0.0257	0.0310	0.0162	0.0088	0.0101
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	-2.0677	-1.9619	-2.0252	-2.0445	-1.9553	-2.0112	-1.9636	-1.9960	-1.9923
Bias	-0.0677	0.0381	-0.0252	-0.0445	0.0447	-0.0112	0.0364	0.0040	0.0077
S.D. $\hat{\lambda}_0$	0.1943	0.1461	0.1661	0.1874	0.1494	0.1644	0.0903	0.0549	0.0574
M.S.E.	0.2039	0.1496	0.1664	0.1917	0.1552	0.1640	0.0965	0.0545	0.0573
$\bar{\lambda}_1$	0.0654	0.0726	0.0580	0.0641	0.0723	0.0578	0.0619	0.0711	0.0673
Bias	-0.0046	0.0026	-0.0120	-0.0059	0.0023	-0.0122	-0.0081	0.0011	-0.0027
S.D. $\hat{\lambda}_1$	0.0290	0.0216	0.0274	0.0296	0.0249	0.0280	0.0133	0.0091	0.0097
M.S.E.	0.0291	0.0215	0.0297	0.0300	0.0249	0.0304	0.0155	0.0091	0.0100

Table 9. Gompertz hazard function -- lightly censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	-2.1238	-2.0200	-2.1039	-2.1076	-2.0242	-2.0977	-2.0561	-2.0526	-2.0604
Bias	-0.1238	-0.0200	-0.1039	-0.1076	-0.0242	-0.0977	-0.0561	-0.0526	-0.0604
S.D.$\hat{\lambda}_0$	0.2468	0.1931	0.2096	0.2239	0.1763	0.1973	0.0605	0.0589	0.0599
M.S.E.	0.2739	0.1922	0.2321	0.2474	0.1771	0.2193	0.0821	0.0785	0.0846
$\bar{\lambda}_1$	0.0586	0.0641	0.0579	0.0604	0.0678	0.0605	0.0619	0.0635	0.0634
Bias	-0.0114	-0.0059	-0.0121	-0.0096	-0.0022	-0.0095	-0.0081	-0.0065	-0.0066
S.D.$\hat{\lambda}_1$	0.0594	0.0480	0.0523	0.0530	0.0425	0.0488	0.0142	0.0135	0.0138
M.S.E.	0.0599	0.0479	0.0532	0.0536	0.0423	0.0495	0.0162	0.0149	0.0152
Actuarial Estimate									
$\bar{\lambda}_0$	-2.0723	-1.9665	-2.0486	-2.0567	-1.9711	-2.0434	-2.0042	-1.9977	-2.0059
Bias	-0.0723	0.0335	-0.0486	-0.0567	0.0289	-0.0434	-0.0042	0.0023	-0.0059
S.D.$\hat{\lambda}_0$	0.2581	0.2047	0.2210	0.2340	0.1868	0.2078	0.0640	0.0628	0.0637
M.S.E.	0.2655	0.2054	0.2241	0.2396	0.1881	0.2113	0.0635	0.0622	0.0633
$\bar{\lambda}_1$	0.0663	0.0705	0.0653	0.0686	0.0747	0.0684	0.0698	0.0709	0.0710
Bias	-0.0037	0.0005	-0.0047	-0.0014	0.0047	-0.0016	-0.0002	0.0009	0.0010
S.D.$\hat{\lambda}_1$	0.0634	0.0516	0.0560	0.0566	0.0458	0.0521	0.0154	0.0147	0.0149
M.S.E.	0.0629	0.0511	-0.0556	0.0563	0.0458	0.0519	0.0152	0.0146	0.0148
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	-2.0715	-1.9655	-2.0476	-2.0559	-1.9702	-2.0425	-2.0034	-1.9968	-2.0049
Bias	-0.0715	0.0345	-0.0476	-0.0559	0.0298	-0.0425	-0.0034	0.0032	-0.0049
S.D.$\hat{\lambda}_0$	0.2588	0.2052	0.2217	0.2345	0.1872	0.2084	0.0643	0.0631	0.0639
M.S.E.	0.2660	0.2060	0.2246	0.2399	0.1886	0.2117	0.0637	0.0625	0.0634
$\bar{\lambda}_1$	0.0669	0.0706	0.0659	0.0692	0.0748	0.0690	0.0703	0.0714	0.0715
Bias	-0.0031	0.0006	-0.0041	-0.0008	0.0048	-0.0010	0.0003	0.0014	0.0015
S.D.$\hat{\lambda}_1$	0.0638	0.0518	0.0563	0.0569	0.0460	0.0523	0.0155	0.0147	0.0150
M.S.E.	0.0632	0.0513	0.0559	0.0566	0.0460	0.0520	0.0153	0.0146	0.0149

Table 10. Gompertz hazard function -- heavily censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	-2.1874	-2.0386	-2.1585	-2.1206	02.0141	-2.1096	-2.0615	-2.0556	-2.0632
Bias	-0.1874	-0.0386	-0.1585	-0.1206	-0.0141	-0.1096	-0.0615	-0.0556	-0.0632
S.D. $\hat{\lambda}_0$	0.3604	0.2890	0.3038	0.3595	0.2761	0.3129	0.0936	0.0867	0.0870
M.S.E.	0.4030	0.2887	0.3400	0.3775	0.2751	0.3301	0.1112	0.1023	0.1068
$\bar{\lambda}_1$	0.1035	0.0667	0.0958	0.0724	0.0504	0.0707	0.0625	0.0620	0.0635
Bias	0.0335	-0.0033	0.0258	0.0024	-0.0196	0.0007	-0.0075	-0.0080	-0.0065
S.D. $\hat{\lambda}_1$	0.1673	0.1326	0.1467	0.1706	0.1352	0.1543	0.0476	0.0436	0.0448
M.S.E.	0.1690	0.1313	0.1475	0.1698	0.1359	0.1535	0.0477	0.0439	0.0448
Actuarial Estimate									
$\bar{\lambda}_0$	-2.1469	-1.9935	-2.1123	-2.0786	-1.9681	-2.0626	-2.0205	-2.0092	-2.0172
Bias	-0.1469	0.0065	-0.1123	-0.0786	0.0319	-0.0626	-0.0205	-0.0092	-0.0172
S.D. $\hat{\lambda}_0$	0.3700	0.3029	0.3152	0.3702	0.2895	0.3253	0.0974	0.0910	0.0912
M.S.E.	0.3946	0.2999	0.3316	0.3766	0.2898	0.3297	0.0986	0.0906	0.0919
$\bar{\lambda}_1$	0.1193	0.0801	0.1098	0.0872	0.0629	0.0838	0.0777	0.0750	0.0769
Bias	0.0493	0.0101	0.0398	0.0172	-0.0071	0.0138	0.0077	0.0050	0.0069
S.D. $\hat{\lambda}_1$	0.1734	0.1404	0.1532	0.1774	0.1429	0.1616	0.0502	0.0463	0.0474
M.S.E.	0.1786	0.1394	0.1568	0.1773	0.1424	0.1614	0.0503	0.0461	0.0474
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	-2.1462	-1.9930	-2.1114	-2.0779	-1.9674	-2.0617	-2.0199	-2.0085	-2.0164
Bias	-0.1462	0.0070	-0.1114	-0.0779	-0.0326	-0.0617	-0.0199	-0.0085	-0.0164
S.D. $\hat{\lambda}_0$	0.3703	0.3035	0.3157	0.3705	0.2901	0.3258	0.0975	0.0912	0.0914
M.S.E.	0.3947	0.3005	0.3318	0.3768	0.2905	0.3300	0.0986	0.0907	0.0920
$\bar{\lambda}_1$	0.1200	0.0807	0.1104	0.0879	0.0634	0.0844	0.0783	0.0756	0.0775
Bias	0.0500	0.0107	0.0404	0.0179	-0.0066	0.0144	0.0083	0.0056	0.0075
S.D. $\hat{\lambda}_1$	0.1737	0.1409	0.1535	0.1777	0.1433	0.1620	0.0503	0.0464	0.0476
M.S.E.	0.1791	0.1399	0.1572	0.1777	0.1427	0.1618	0.0505	0.0463	0.0477

Table 11. Weibull hazard function -- complete data

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	0.0827	0.0869	0.0782	0.0800	0.0845	0.0770	0.0800	0.0773	0.0753
Bias	0.0027	0.0069	-0.0018	0.0	0.0045	-0.0030	0.0	-0.0027	-0.0047
S.D.$\hat{\lambda}_0$	0.0344	0.0381	0.0239	0.0279	0.0296	0.0216	0.0095	0.0056	0.0068
M.S.E.	0.0342	0.0383	0.0237	0.0278	0.0298	0.0217	0.0094	0.0062	0.0082
$\bar{\lambda}_1$	1.3999	1.4526	1.4324	1.4081	1.4562	1.4371	1.4193	1.4596	1.4675
Bias	-0.1001	-0.0474	-0.0676	-0.0919	-0.0438	-0.0629	-0.0807	-0.0404	-0.0325
S.D.$\hat{\lambda}_1$	0.1566	0.1460	0.1364	0.1406	0.1272	0.1335	0.0645	0.0340	0.0429
M.S.E.	0.1845	0.1521	0.1510	0.1674	0.1339	0.1470	0.1029	0.0526	0.0535
Actuarial Estimate									
$\bar{\lambda}_0$	0.0837	0.0854	0.0801	0.0808	0.0834	0.0789	0.0812	0.0785	0.0776
Bias	0.0037	0.0054	0.0001	0.0008	0.0034	-0.0011	0.0012	-0.0015	-0.0024
S.D.$\hat{\lambda}_0$	0.0345	0.0301	0.0242	0.0280	0.0244	0.0222	0.0099	0.0061	0.0072
M.S.E.	0.0344	0.0303	0.0240	0.0279	0.0245	0.0221	0.0099	0.0062	0.0075
$\bar{\lambda}_1$	1.4590	1.5112	1.4801	1.4677	1.5160	1.4847	1.4755	1.5099	1.5102
Bias	-0.0410	0.0112	-0.0199	-0.0323	0.0160	-0.0153	-0.0245	0.0099	0.0102
S.D.$\hat{\lambda}_1$	0.1612	0.1314	0.1366	0.1451	0.1192	0.1355	0.0692	0.0358	0.0441
M.S.E.	0.1648	0.1306	0.1367	0.1479	0.1197	0.1357	0.0728	0.0368	0.0448
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	0.0829	0.0848	0.0797	0.0802	0.0830	0.0786	0.0806	0.0784	0.0776
Bias	0.0029	0.0048	-0.0003	0.0002	0.0030	-0.0014	0.0006	-0.0016	-0.0024
S.D.$\hat{\lambda}_0$	0.0333	0.0270	0.0237	0.0273	0.0225	0.0218	0.0098	0.0061	0.0072
M.S.E.	0.0331	0.0272	0.0235	0.0272	0.0226	0.0217	0.0097	0.0062	0.0075
$\bar{\lambda}_1$	1.4685	1.5124	1.4859	1.4772	1.5174	1.4903	1.4839	1.5134	1.5137
Bias	-0.0315	0.0124	-0.0141	-0.0228	0.0174	-0.0097	-0.0161	0.0134	0.0137
S.D.$\hat{\lambda}_1$	0.1617	0.1235	0.1358	0.1457	0.1139	0.1350	0.0698	0.0360	0.0441
M.S.E.	0.1631	0.1229	0.1352	0.1468	0.1147	0.1347	0.0709	0.0381	0.0458

Table 12. Weibull hazard function -- lightly censored

	R=50 N =100			R=100 N =100			R=50 N =1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	0.0745	0.0824	0.0733	0.0744	0.0811	0.0735	0.0742	0.0768	0.0745
Bias	-0.0055	0.0024	-0.0067	-0.0056	0.0011	-0.0065	-0.0058	-0.0032	-0.0055
S.D. $\hat{\lambda}_0$	0.0220	0.0221	0.0184	0.0217	0.0206	0.0195	0.0079	0.0060	0.0070
M.S.E.	0.0225	0.0220	0.0194	0.0223	0.0205	0.0205	0.0097	0.0067	0.0088
$\bar{\lambda}_1$	1.4710	1.4623	1.4851	1.4699	1.4699	1.4845	1.4774	1.4642	1.4787
Bias	-0.0290	-0.0377	-0.0149	-0.0301	-0.0301	-0.0155	-0.0226	-0.0358	-0.0213
S.D. $\hat{\lambda}_1$	0.1458	0.1300	0.1370	0.1575	0.1326	0.1504	0.0507	0.0374	0.0475
M.S.E.	0.1472	0.1341	0.1364	0.1596	0.1353	0.1504	0.0550	0.0515	0.0516
Actual Estimate									
$\bar{\lambda}_0$	0.0774	0.0848	0.0760	0.0772	0.0833	0.0761	0.0769	0.0789	0.0772
Bias	-0.0026	0.0048	-0.0040	-0.0028	0.0033	-0.0039	-0.0031	-0.0011	-0.0028
S.D. $\hat{\lambda}_0$	0.0234	0.0231	0.0195	0.0230	0.0218	0.0207	0.0082	0.0065	0.0074
M.S.E.	0.0233	0.0234	0.0197	0.0231	0.0219	0.0210	0.0087	0.0065	0.0078
$\bar{\lambda}_1$	1.5090	1.5024	1.5238	1.5084	1.5113	1.5235	1.5164	1.5070	1.5172
Bias	0.0090	0.0024	0.0238	0.0084	0.0113	0.0235	0.0164	0.0070	0.0172
S.D. $\hat{\lambda}_1$	0.1506	0.1355	0.1415	0.1642	0.1412	0.1563	0.0518	0.0400	0.0485
M.S.E.	0.1494	0.1342	0.1421	0.1636	0.1409	0.0573	0.0538	0.0402	0.0510
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	0.0774	0.0847	0.0760	0.0772	0.0833	0.0761	0.0769	0.0788	0.0772
Bias	-0.0026	0.0047	-0.0040	-0.0028	0.0033	-0.0039	-0.0031	-0.0012	-0.0028
S.D. $\hat{\lambda}_0$	0.0234	0.0230	0.0195	0.0230	0.0218	0.0207	0.0082	0.0065	0.0074
M.S.E.	0.0233	0.0232	0.0197	0.0231	0.0219	0.0210	0.0087	0.0065	0.0078
$\bar{\lambda}_1$	1.5116	1.5039	1.5264	1.5111	1.5130	1.5261	1.5188	1.5098	1.5195
Bias	0.0116	0.0039	0.0264	0.0111	0.0130	0.0261	0.0188	0.0098	0.0195
S.D. $\hat{\lambda}_1$	0.1510	0.1349	0.1419	0.1648	0.1414	0.1568	0.0519	0.0402	0.0486
M.S.E.	0.1499	0.1336	0.1429	0.1643	0.1413	0.1582	0.0547	0.0410	0.0519

Table 13. Weibull hazard function -- heavily censored

	R=50 N_1=100			R=100 N_1=100			R=50 N_1=1000		
	W1	W2	W3	W1	W2	W3	W1	W2	W3
Conditional Proportion Retired									
$\bar{\lambda}_0$	0.0730	0.0813	0.0724	0.0732	0.0788	0.0725	0.0739	0.0751	0.0740
Bias	-0.0070	0.0013	-0.0076	-0.0068	-0.0012	-0.0075	-0.0061	-0.0049	-0.0060
S.D.$\hat{\lambda}_0$	0.0217	0.0231	0.0199	0.0221	0.0224	0.0210	0.0081	0.0067	0.0075
M.S.E.	0.0226	0.0229	0.0211	0.0230	0.0223	0.0222	0.0101	0.0082	0.0095
$\bar{\lambda}_1$	1.5160	1.4645	1.5221	1.5191	1.4976	1.5276	1.4984	1.4877	1.4977
Bias	0.0160	-0.0355	0.0221	0.0191	-0.0024	0.0276	-0.0016	-0.0123	-0.0023
S.D.$\hat{\lambda}_1$	0.2489	0.2234	0.2377	0.2385	0.2171	0.2316	0.0762	0.0625	0.0726
M.S.E.	0.2469	0.2240	0.2363	0.2381	0.2160	0.2321	0.0755	0.0631	0.0719
Actuarial Estimate									
$\bar{\lambda}_0$	0.0758	0.0839	0.0752	0.0761	0.0814	0.0753	0.0767	0.0778	0.0768
Bias	-0.0042	0.0039	-0.0048	-0.0039	0.0014	-0.0047	-0.0033	-0.0022	-0.0032
S.D.$\hat{\lambda}_0$	0.0227	0.0242	0.0209	0.0233	0.0237	0.0222	0.0085	0.0072	0.0079
M.S.E.	0.0229	0.0243	0.0212	0.0235	0.0236	0.0226	0.0090	0.0075	0.0084
$\bar{\lambda}_1$	1.5503	1.5035	1.5571	1.5538	1.5363	1.5632	1.5320	1.5235	1.5317
Bias	0.0503	0.0035	0.0571	0.0538	0.0363	0.0632	0.0320	0.0235	0.0317
S.D.$\hat{\lambda}_1$	0.2582	0.2379	0.2481	0.2462	0.2297	0.2403	0.0773	0.0655	0.0739
M.S.E.	0.2605	0.2355	0.2522	0.2508	0.2314	0.2473	0.0829	0.0690	0.0797
Maximum Likelihood Estimate									
$\bar{\lambda}_0$	0.0758	0.0838	0.0752	0.0761	0.0814	0.0753	0.0767	0.0778	0.0768
Bias	-0.0042	0.0038	-0.0048	-0.0039	0.0014	-0.0047	-0.0033	-0.0022	-0.0032
S.D.$\hat{\lambda}_0$	0.0227	0.0242	0.0210	0.0233	0.0237	0.0222	0.0085	0.0072	0.0079
M.S.E.	0.0229	0.0243	0.0213	0.0235	0.0236	0.0226	0.0090	0.0075	0.0084
$\bar{\lambda}_1$	1.5518	1.5055	1.5587	1.5554	1.5381	1.5649	1.5334	1.5252	1.5331
Bias	0.0518	0.0055	0.0587	0.0554	0.0381	0.0649	0.0334	0.0252	0.0331
S.D.$\hat{\lambda}_1$	0.2589	0.2393	0.2490	0.2467	0.2307	0.2409	0.0774	0.0657	0.0740
M.S.E.	0.2615	0.2370	0.2534	0.2516	0.2327	0.2483	0.0836	0.0698	0.0804

containing either 100 or 1000 units per vintage. The number of vintages included in a study of 100 units per vintage was increased from 50 to 100 in order to determine whether or not 50 replications was sufficient to estimate the underlying population parameters. Thus, the notation $R = 50$, $N_1 = 100$ describes an analysis of 50 vintages (replications) containing 100 units per vintage.

It was noted earlier that the vector of observed hazard rates Y was weighted by either 1.0 (i.e., no weighting), $1/\widehat{Var}(\hat{\lambda}_k)$, or $N_k h_k$ to obtain a weighted least squares estimate of the parameters of the underlying hazard function. The estimates obtained from each of these weights are identified in Tables 2 thru 13 as W1, W2, W3; where W1 is an unweighted estimate, W2 is weighting by the inverse of the estimated variance of the hazard rate, and W3 is weighting by the number of units entering an age-interval times the width of the interval.

The rows of Tables 2 thru 13 are divided into three major sections which identify the parameter estimates and related statistics associated with (1) the conditional proportion retired, (2) the actuarial estimate, and (3) the maximum likelihood estimate of the hazard rate for each age-interval. The statistics computed for a given model, estimator, vintage size, number of replications, and weighting are defined as follows:

$$\bar{\lambda}_j = \frac{1}{R} \sum_{i=1}^{R} \hat{\lambda}_{ji} \; ; \qquad j = 0, 1$$

$$\text{Bias} = \bar{\lambda}_j - \lambda_j \; ; \qquad j = 0, 1$$

$$\text{S.D.}\hat{\lambda}_j = \sqrt{\frac{\sum_{i=1}^{R} (\hat{\lambda}_{ji} - \bar{\lambda}_j)^2}{R - 1}} \quad ; \qquad j = 0, 1$$

$$\text{M.S.E.} = \sqrt{\frac{R - 1}{R} (\text{S.D.}\hat{\lambda}_j)^2 + \text{Bias}^2} \quad ; \qquad j = 0, 1.$$

In words, $\bar{\lambda}_j$ is approximately the mean or average of the probability distribution of the estimator $\hat{\lambda}_j$ ($j = 0, 1$). The Bias is the difference between the mean of the probability distribution of the estimator and the true value of λ_j -- the population parameter of the underlying distribution. The standard deviation, S.D.$\hat{\lambda}_j$, is calculated as the square root of the sum of the deviations squared divided by the number of vintages less one. It should be noted that the mean square error (M.S.E.) is usually defined as the sum of the population variance and the bias squared. The statistic shown in Tables 2 thru 13 is the square root of this quantity or, more properly defined, the root mean square error.

The results shown in Table 2 were derived from a constant hazard function which has a probability density function f(t) and a survivorship function S(t) that are negative exponential. The simplicity of this model (i.e., a single parameter) offers the possibility of a reasonably good analysis of the statistical properties of the weighted and unweighted estimates of the hazard rate. The consistency of the results also suggests that the exponential distribution is well-suited to a comparative analysis of the properties of the estimators.

It is evident from Table 2 that the maximum likelihood estimate, weighted by the number of units entering an age-interval, consistently yields an estimate of λ_0 that is closer to the true value (i.e., 0.20) than either the conditional proportion retired or the actuarial estimate. The reasonableness of this result can be verified by calculating the theoretical bias of each estimator from the equations developed for the expected value of $\hat{\lambda}_k$. The magnitude of the theoretical bias should be comparable to the unweighted bias shown in Table 2.

The theoretical bias of the conditional proportion retired can be calculated from Equation 18 where, under a constant hazard function, it can be shown that

$$q_k = (1 - e^{-\alpha h_k})$$

and

$$E[\hat{\lambda}_k] = \frac{1}{h_k}(1 - e^{-\alpha h_k})$$

where $\alpha = \lambda(t)$. Thus, when $\alpha = 0.20$ and $h_k = 1$,

$$E[\hat{\lambda}_k] = 1 - e^{-0.20} = 0.1813$$

and the theoretical bias becomes

$$\begin{aligned} \text{Bias} &= E[\hat{\lambda}_k] - \lambda_k \\ &= 0.1813 - 0.20 \\ &= -0.0187. \end{aligned}$$

Similarly, the theoretical bias of the actuarial estimate can be calculated from Equation 20 where, under a constant hazard function, it can be shown that

$$q_k = (1 - e^{-\alpha h_k}),$$

$$(1 - \phi_k) = e^{-\alpha t_k}$$

and

$$E[\hat{\lambda}_k] = \frac{(1 - e^{-\alpha h_k})}{h_k[1 - \tfrac{1}{2}(1 - e^{-\alpha h_k})]} \left\{ 1 + \frac{e^{-\alpha h_k}(1 - e^{-\alpha h_k})}{4[1 - \tfrac{1}{2}(1 - e^{-\alpha h_k})]^2} + \frac{e^{-\alpha h_k}}{2[1 - \tfrac{1}{2}(1 - e^{-\alpha h_k})]} \left[\frac{1}{N_1 e^{-\alpha t_k}} \right] \right\} \tag{31}$$

where $\alpha = \lambda(t)$. Thus, when $\alpha = 0.20$ and $h_k = 1$,

$$E[\hat{\lambda}_k] = 0.2083$$

which is obtained by evaluating only those terms of Equation 31 which do not depend on N_1, the vintage size. The theoretical bias then becomes

$$\begin{aligned} \text{Bias} &= E[\lambda_k] - \lambda_k \\ &= 0.2083 - 0.20 \\ &= 0.0083. \end{aligned}$$

The calculation of the bias of the maximum likelihood estimate is rather complicated since, under a constant hazard function, an evaluation of Equation 28 yields an expression of the form

$$E[\hat{\lambda}_k] = \alpha + \frac{1 - e^{-\alpha}}{2N_1 e^{-\alpha t_{k+1}}} \qquad (32)$$

where the second term of the right-hand side of Equation 32 is the bias. Thus, the bias of the maximum likelihood estimate is a function of both N_1, the vintage size, and t_{k+1}, the end point of the k^{th} age-interval. An example of the bias was calculated, however, by evaluating Equation 32 for $\alpha = 0.20$, $N_1 = 100$, and $t_{k+1} = 4$. The resulting bias is 0.002.

Thus, the theoretical bias of the maximum likelihood estimate is less than the bias of either the conditional proportion retired or the actuarial estimate, which is consistent with the results shown in Table 2. This is not totally surprising, however, since it can be shown that the maximum likelihood estimate is asymptotically unbiased for large values of N_1. It should also be noted that the maximum likelihood estimate of λ_k (i.e., Equation 25) was developed under the assumption that a hazard function is constant within each age-interval.[1] The exponential distribution is consistent with this assumption and should, therefore, improve the relative bias of the maximum likelihood estimate.

Although an unbiased estimator is generally preferred over a biased estimator, unbiasedness is not necessarily an indispensable property of a "good" estimator. If the amount of bias is small compared with the

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[1] Supra, p. 52.

standard deviation of the estimator, the estimator though biased may be entirely satisfactory. It is important, therefore, to also consider the standard deviation of the estimates obtained from each estimate of the hazard rate.

It is evident from Table 2 that the conditional proportion retired, weighted by the number of units entering an age-interval, consistently yields a smaller standard deviation of $\hat{\lambda}_0$ than either the actuarial estimate or the maximum likelihood estimate. This result is not totally satisfying, however, since one is now confronted with the problem of choosing between an estimator that yields a relatively small bias (i.e., the maximum likelihood estimate) and an estimator that yields a relatively small standard deviation (i.e., the conditional proportion retired). It is helpful, therefore, to combine the bias and standard deviation into a single statistic which provides a joint measurement of the two properties. The root mean square error has been used for this purpose.

The analysis shown in Table 2 indicates that the actuarial estimate, weighted by the number of units entering an age-interval, consistently yields a smaller root mean square error than either the conditional proportion retired or the maximum likelihood estimate.

Thus, it can be concluded from Table 2 that each of the three estimators exhibits certain characteristics of a "good" estimator and the choice of which estimator is "best" depends on which statistical property is considered most important. If the underlying hazard function is known to be a constant and a small bias is crucial, then the maximum likelihood estimate should be selected. On the other hand, if a small standard deviation is crucial, then the conditional proportion retired should be selected. If the smallest combined standard deviation and bias is important, then the actuarial estimate should be selected. In all cases, however, weighting by the number of units entering an age-interval is better than weighting by either the inverse of the estimated variance of the hazard rate or an unweighted estimate.

The conclusions drawn from Table 2 are generally applicable to Tables 3 and 4 which provide an analysis of two levels of censoring when the underlying hazard function is known to be a constant. As the data become more censored, however, the bias of the maximum likelihood estimate tends to exceed the bias of the actuarial estimate which is less than the bias of the conditional proportion retired. The conditional proportion retired appears to yield the smallest standard deviation regardless of the degree of censoring.

The results shown in Table 5 were derived from a linear hazard function which necessitates the estimation of two parameters. This slightly more complicated model also contradicts the assumption of a constant hazard function within each age-interval which was postulated to develop the maximum likelihood estimator. It is not surprising, therefore, that the maximum likelihood estimate, weighted by the number of units entering an age-interval, consistently yields a larger bias than the actuarial estimate and a smaller bias than the conditional proportion retired. This result appears to hold for estimates of both λ_0 and λ_1.

It is also evident from Table 5 that the conditional proportion retired, weighted by the number of units entering an age-interval, consistently yields a smaller estimate of the standard deviation of both $\hat{\lambda}_0$ and $\hat{\lambda}_1$ than either the actuarial estimate or the maximum likelihood estimate. It is disconcerting to note, however, that the conditional proportion retired, weighted by the number of units entering an age-interval, consistently yields the smallest root mean square error of $\hat{\lambda}_0$ while the actuarial estimate, weighted by the number of units entering an age-interval, consistently yields the smallest root mean square error of $\hat{\lambda}_1$. Thus, the root mean square error offers little guidance in selecting the "best" estimator for the linear model.

Unlike the constant hazard function, the linear model tends to show a disproportionate change in the bias and standard deviation when the number of replications is increased from 50 to 100. This suggests that the number of vintages included in the study may be insufficient to estimate the population parameters. However, an additional analysis of the linear model which included 500 replications showed no significant change in the bias and standard deviation from the results obtained using 100 replications. Therefore, it is reasonable to conclude that 100 replications is sufficient to estimate the population parameters of the linear model.

The results shown in Tables 6 and 7 suggest that censoring a linear model has a greater effect on the parameter estimates than censoring a constant hazard function. As the data become more censored, the bias, standard deviation, and root mean square error for the actuarial estimate approach the value of the corresponding statistics for the maximum likelihood estimate. This result only holds for estimates of λ_0. Furthermore, as the data become more censored, the conditional proportion retired, weighted by the number of units entering an age-interval, consistently yields the smallest root mean square error for both $\hat{\lambda}_0$ and $\hat{\lambda}_1$.

The results shown in Table 8 were derived from a Gompertz hazard function which also necessitates the estimation of two parameters. The complexity of this model appears to introduce several inconsistencies that were not observed with the previous models. For example, the smallest bias, standard deviation, and mean square error all occur when the estimates are weighted by the inverse of the estimated variance of the hazard rate. The previous models showed the number of units entering an age-interval to be the best weighting. The Gompertz model also yields erratic results as the vintage size is increased from 100 to 1000.

Progressive censoring of the Gompertz model does, however, introduce some consistency in the estimates. The results shown in Tables 9 and 10 suggest that the conditional proportion retired, weighted by the inverse of the estimated variance of the hazard rate, consistently yields the smallest standard deviation and mean standard error for both $\hat{\lambda}_0$ and $\hat{\lambda}_1$. There is not, however, an estimator that consistently yields the smallest bias when the model is censored.

The results shown in Table 11 were derived from a two-parameter Weibull hazard function. There are few, if any, consistencies derived from this model. The smallest bias, standard deviation, and root mean square error are scattered among the estimators as well as among the three methods of weighting. As the data become more censored, however, the results tend to show some regularity. Tables 12 and 13 show that the conditional proportion retired, weighted by either the number of units entering an age-interval or the inverse of the estimated variance of the hazard rate, consistently yields the smallest bias, standard deviation, and root mean square error for both $\hat{\lambda}_0$ and $\hat{\lambda}_1$. It is interesting to note that a censored Weibull model also forces the actuarial and maximum likelihood estimates of

λ_0 to the same value. This tendency was observed in the linear model but did not occur with a Gompertz hazard function.

The results summarized in Tables 2 thru 13 suggest that the bias of the maximum likelihood estimate tends to increase as the underlying hazard function departs from the assumption of a constant hazard rate within each age-interval. It would seem, therefore, that the bias of the maximum likelihood estimate should improve as the average service life increases and the width of an age-interval becomes small in relation to the maximum life of a property unit. This theory was tested with a linear hazard function containing population parameters of $\lambda_0 = 0.1$ and $\lambda_1 = 0.01$. The average service life of this model is approximately 12.5 years which is over twice the average service life of the model used in Table 5. The results of this experiment showed no significant improvement in the bias of the maximum likelihood estimate.

SUMMARY AND CONCLUSIONS

The procedure used to estimate the parameters of a hazard function in life studies of industrial property has traditionally relied on the conditional proportion retired (or retirement ratio) as an estimate of the hazard rate for each age-interval. This so-called "actuarial method" can be viewed as a two-stage procedure in which estimates of the hazard rate are obtained from an observed life table and then used as the dependent variable in a weighted regression analysis to estimate the parameters of an assumed hazard function.

In this study, three different estimates of the hazard rate were developed by nonparametric methods and compared in a Monte Carlo study to determine which estimator and method of weighting is best for depreciation applications. The major conclusions drawn from this investigation are as follows:

(i) The conditional proportion retired, the actuarial estimate, and the maximum likelihood estimate each possess different attributes of a "good" estimator. However, it is difficult to say which attribute is the most important or which estimator is best for depreciation applications.

(ii) The conditional proportion retired tends to yield the smallest standard deviation of the estimated parameters regardless of the form of the underlying hazard function.

(iii) The actuarial estimate tends to yield the smallest root mean square error of the estimated parameters when the sample size is large and the data are uncensored.

(iv) The maximum likelihood estimate tends to yield the smallest bias of the estimated parameters when the form of the underlying hazard function does not significantly violate the assumption of a constant hazard rate within each age-interval.

(v) The conditional proportion retired tends to yield the smallest bias, standard deviation, and root mean square error of the estimated parameters when the data are heavily censored.

(vi) The best method of weighting appears to depend on the form of the underlying hazard function. Weighting by the number of units entering an age-interval times the width of the interval is best when the form of the underlying hazard function is a constant or a polynomial of the first degree. Weighting by the inverse of the estimated variance of the hazard rate is best when the form of the underlying hazard function is a Weibull distribution. The best method of weighting is indeterminate when the form of the underlying hazard function is a Gompertz distribution.

The conclusions drawn from this study raise a number of interesting questions that may warrant further investigation. For example, it was found that the maximum likelihood estimator provides a reasonably good estimate of the population parameters when the form of the underlying hazard function does not significantly violate the assumption of a constant hazard rate within each age-interval. It is possible that this assumption could be met if the age-intervals are small in relation to the average service life of the units installed at age zero. It would be interesting, therefore, to repeat this investigation for average service lives in the range of 20 to 40 years and observe the statistical properties of each estimator and method of weighting as a function of the average service life.

It was also found that the estimators are reasonably consistent when the form of the underlying hazard function is a constant or a polynomial of the first degree. It would be interesting to generalize this model to include quadratic and higher terms. Thus, one might consider a model of the form

$$\lambda(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \ldots + \lambda_m t^m$$

which is commonly used in life studies of industrial property when the form of the underlying hazard function is assumed to follow the Iowa-type survival functions. It may be that subsequent fitting of the smoothed survivorship function to the Iowa curves would introduce a different criterion for measuring the statistical properties of the estimators. In this connection, an attempt was made to fit first, second, and third degree polynomials to the three estimates of the hazard rate followed by fitting the smoothed survivorship function to the Iowa curves. The results suggested that the actuarial estimate and the maximum likelihood estimate may yield a shorter average service life than the conditional proportion retired.

Finally, it should be emphasized that the end result of life analysis is the estimation of a proper depreciation accrual rate based upon engineering judgement of events likely to occur in the future. This suggests that one should not go too far in attempts to polish statistical methods; the effort may exceed the usefulness of the results.

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ACKNOWLEDGMENTS

A number of people were instrumental in influencing the research and writing of this dissertation. I am deeply indebted to Dr. H. A. Cowles for his guidance and support as my major professor and chairman of the examining committee. A special acknowledgment is due Dr. R. W. Mensing who, in addition to serving on the examining committee, devoted many hours in reading the preliminary drafts and assisting with the statistical developments. I am also grateful to Dr. L. B. Fletcher, Dr. K. L. McRoberts, and Dr. G. W. Smith who willingly served as members of the examining committee.

A special word of thanks is extended to Mr. M. L. Currier who assisted in designing and executing the computer programs used in this study. The financial assistance and support received from Northern States Power Company was also instrumental in making this study possible.

Last, but certainly not least, I am indebted to my wife Joyce, not only for typing the preliminary and final drafts, but also for her patience, understanding, and frequent words of encouragement.

APPENDIX A: DERIVATION OF THE h-SYSTEM OF SURVIVAL FUNCTIONS

Consider the function

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\{-t^2/2\}; \qquad -\infty < t < \infty \tag{33}$$

which is the well-known "normal" probability density function (p.d.f.) of a random variable T with mean $\mu = 0$ and variance $\sigma^2 = 1.0$. Clearly, since $\phi(t)$ is defined over a range which includes values of t approaching $-\infty$, $\phi(t)$ cannot be used to describe the probability distribution of the service life of a unit of property.

It is a simple matter, however, to construct a linear transformation of t and truncate a portion of $\phi(t)$ such that the transformed variable describes the service life of an asset and the portion of $\phi(t)$ remaining after truncation satisfies the properties of a density function. This construction can be visualized from Figure 1 which shows $\phi(t)$ truncated at some arbitrary distance h from t = 0.

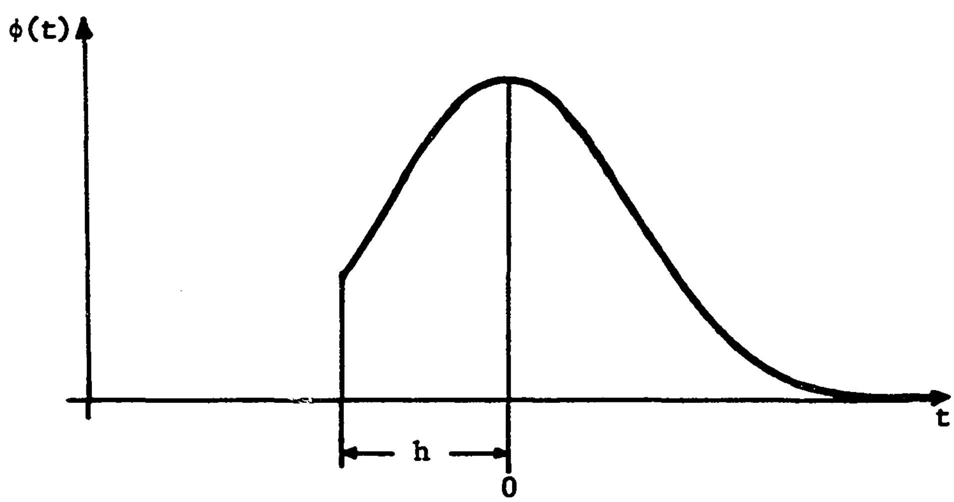

Figure 1. A truncated standard normal density function.

This point of truncation can be related to the service life of an asset by letting $t = -h$ represent the point in time at which a unit of property is installed. By definition, $t = -h$ is taken to be age zero. Thus, $T' = T + h$ can be defined as a new random variable with a p.d.f. given by the portion of $\phi(t)$ remaining after truncation. The mean or expected value of T' is easily obtained by letting

$$\Phi(-h) = \int_{-h}^{\infty} \phi(t)dt \tag{34}$$

which is simply the area under the portion of $\phi(t)$ remaining after truncation, and calculating the first moment of T' about $t = -h$. Thus, using Kimball's (42) notation for the expected value of T', we obtain

$$E[T'] = w = \frac{\int_{-h}^{\infty} t'\phi(t)dt}{\int_{-h}^{\infty} \phi(t)dt} = \frac{\int_{-h}^{\infty} (t+h)\phi(t)dt}{\Phi(-h)}$$

$$= \frac{\int_{-h}^{\infty} t\phi(t)dt + h\int_{-h}^{\infty} \phi(t)dt}{\Phi(-h)}$$

$$= \frac{\int_{-h}^{\infty} t\phi(t)dt + h\Phi(-h)}{\Phi(-h)} . \tag{35}$$

Now, using Equation 33 we can write

$$\int_{-h}^{\infty} t\phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-h}^{\infty} t \exp\{-t^2/2\}dt$$

which is easily evaluated by letting

$$z = t^2/2, \qquad dz = tdt$$

and noting that $z = h^2/2$ when $t = -h$. Thus,

$$\int_{-h}^{\infty} t\phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{h^2/2}^{\infty} \exp\{-z\}dz$$

$$= \frac{1}{\sqrt{2\pi}} \exp\{-h^2/2\}$$

$$= \phi(-h).$$

Using this result with Equation 35 we obtain for the mean of the truncated distribution

$$w = \frac{\phi(-h) + h\Phi(-h)}{\Phi(-h)} = \frac{\phi(-h)}{\Phi(-h)} + h. \tag{36}$$

The location of w in relation to t = 0 can be visualized from Figure 2 which also shows the location of w in terms of a new random variable T'/w.

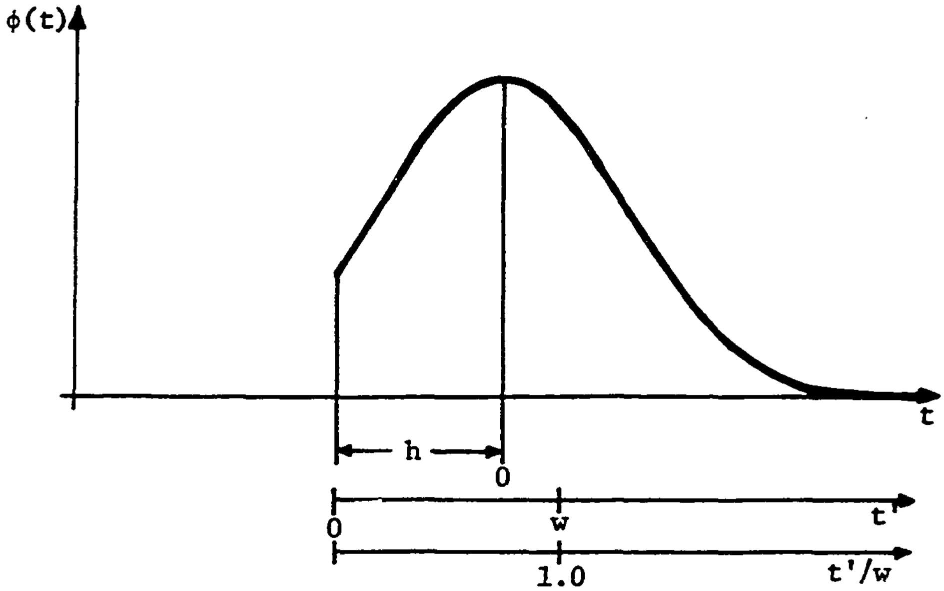

Figure 2. Relationship between w and various transformations of t.

Our motivation for constructing T'/w becomes apparent when we observe that the mean or expected value of T'/w is 1.0. In other words,

$$E[T'/w] = \frac{\int_{-h}^{\infty} \frac{t'}{w} \phi(t)dt}{\int_{-h}^{\infty} \phi(t)dt} = \frac{\frac{1}{w}\int_{-h}^{\infty} t'\phi(t)dt}{\int_{-h}^{\infty} \phi(t)dt}$$

$$= \frac{1}{w}(w)$$

$$= 1.0$$

which is precisely the result we would obtain if t'/w was taken to represent the service life of an asset divided by its life expectancy at age zero. This relationship can be expressed in terms of t by letting x represent the service life of an asset (i.e., the age of an asset when it is retired from service) and L represent its life expectancy at age zero (i.e., average service life). Then, by definition,

$$\frac{t'}{w} = \frac{x}{L}$$

from which it follows that

$$\frac{t+h}{w} = \frac{x}{L}$$

and

$$t = w(x/L) - h. \tag{37}$$

Now, from our previous use of Equation 33 and 34 it should be clear that the p.d.f. of T for $\phi(t)$ truncated at $t = -h$ can be written as

$$f(t) = \frac{\phi(t)}{\Phi(-h)}\,; \qquad -h \leq t < \infty. \tag{38}$$

From Equation 38 it follows that the probability $Pr[T > t]$, which we denote by $S(t)$, is given by

$$S(t) = Pr[T > t] = 1 - \int_{-h}^{t} f(s)ds = \int_{t}^{\infty} f(s)ds$$

$$= \frac{1}{\Phi(-h)} \int_{t}^{\infty} \phi(s)ds = \frac{\Phi(t)}{\Phi(-h)}\,; \qquad -h \leq t < \infty. \tag{39}$$

Equation 39 is, of course, the probability statement used in Equation 9 to define a survivorship function. We can, therefore, use Equation 37 as an expression for t and write the probability that a unit of property survives (i.e., remains in service) beyond age x as

$$S(x) = \frac{\Phi(wx/L - h)}{\Phi(-h)}; \qquad 0 \leq x < \infty. \tag{40}$$

Thus, Equation 40 defines a two parameter distribution which describes the h-System survivorship function. The general shape of this function for various values of h can be visualized from Figure 3, which has been reproduced from Kimball (42).

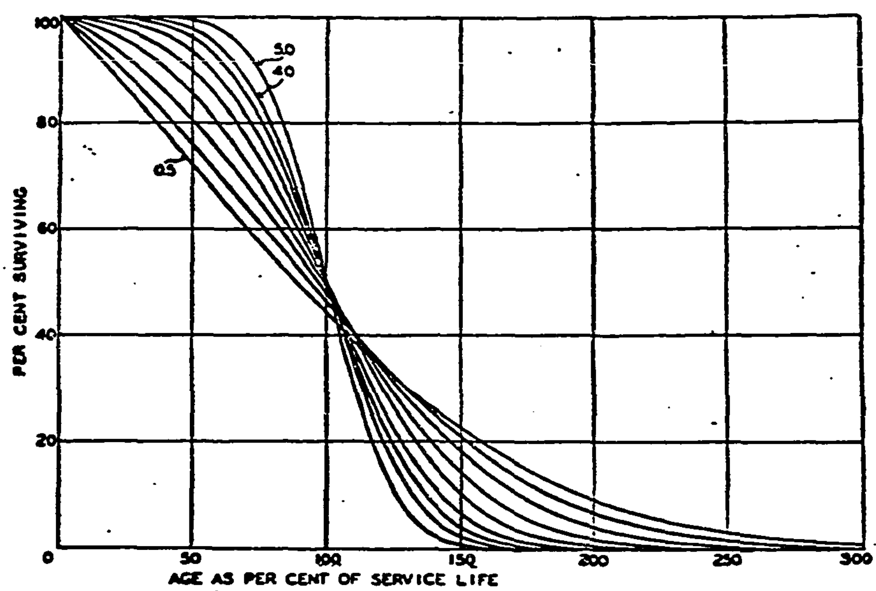

Figure 3. h-System survivorship functions.

The relationship between a retirement frequency function f(x) and a survivorship function S(x) is given by Equation 8 and 9, i.e.,

$$f(x) = \frac{-dS(x)}{dx}$$

which, for the h-System becomes

$$f(x) = \frac{-d\Phi(wx/L - h)}{\Phi(-h)dx}$$

$$= \frac{w\phi(wx/L - h)}{L\Phi(-h)} \qquad (41)$$

The general shape of the function given by Equation 41 for various values of h is illustrated in Figure 4, which has also been reproduced from Kimball (42).

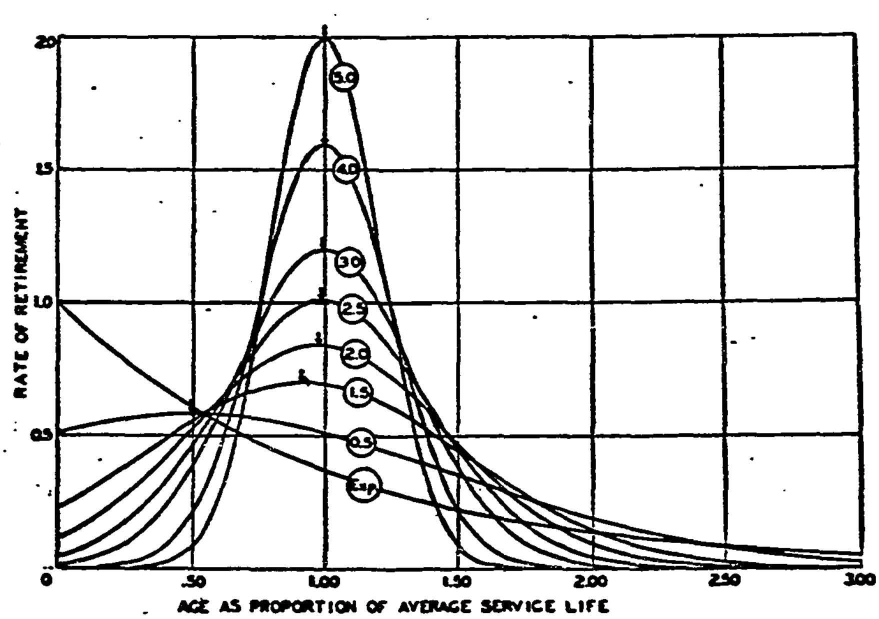

Figure 4. h-System retirement frequency functions.

Thus, a life table for the h-System can be generated from Equation 41 by evaluating

$$\frac{w\int_{x_1}^{x_2} \phi(wx/L - h)dx}{L\Phi(-h)}$$

for each age-interval where x_1 and x_2 denote age at the beginning and end of a given interval. This calculation has been computerized (61) using Simpson's Rule to evaluate the integral and a table of the normal probability function to obtain $\Phi(-h)$.

APPENDIX B: PROGRAM LISTING OF THE ACTUARIAL METHOD OF LIFE ANALYSIS

```
C
C         PROGRAM IDENTIFICATION
C
C                     *****************************************
C                     *  ACTUARIAL METHOD OF LIFE ANALYSIS  *
C                     *     WRITTEN BY A.D. KENNEDY  7/70     *
C                     *    REVISED BY A.D. KENNEDY 10/70      *
C                     *         TEXAS MEDICAL CENTER          *
C                     *    REVISED BY R.E. WHITE   12/76      *
C                     *    NORTHERN STATES POWER COMPANY      *
C                     *****************************************
C
C         CARD INPUT FORMAT - TWO TYPES OF DATA CARDS ARE REQUIRED FOR
C         EACH SET OF RETIREMENT DATA.  ANY NUMBER OF ANALYSES MAY BE
C         RUN FOR EACH SET OF DATA AND ANY NUMBER OF SETS OF DATA ARE
C         ALLOWABLE.
C
C         (1) ANALYSIS INFORMATION CARD
C             CC. 1-8    ACNTNO  - A UNIQUE 8-CHARACTER ALPHAMERIC NAME
C                                  OR NUMBER ASSIGNED TO EACH DIFFERENT
C                                  SET OF RETIREMENT DATA, (A8).
C             CC. 61-63  JMAX    - NUMBER OF AGE-INTERVALS, 0 LT. JMAX
C                                  LE. 100, (I3).
C             CC. 66-74  XENT    - NUMBER OF UNITS ENTERING THE INI-
C                                  TIAL AGE-INTERVAL, (F9.0).
C             CC. 77     METHOD  - METHOD USED TO ESTIMATE HAZARD RATE.
C                                  1 - CONDITIONAL PROPORTION RETIRED.
C                                  2 - ACTUARIAL ESTIMATE.
C                                  3 - MAXIMUM LIKELIHOOD ESTIMATE.
C         (2) INTERVAL DATA CARD - A SET OF INTERVAL DATA CARDS FOLLOWS
C             THE ANALYSIS INFORMATION CARD FOR THE FIRST ANALYSIS OF
C             EACH SET OF RETIREMENT DATA.  THE PROGRAM REQUIRES ONE
C             INTERVAL DATA CARD FOR EACH AGE INTERVAL (I.E., 'JMAX'
C             ARE REQUIRED).
C             CC. 11-19 TINTV(I) - LOWER LIMIT OF THE I-TH TIME
C                         INTERVAL, TINTV(I) GE. 0, F(9.2).
C             CC. 22-30  XDI(I)  - NUMBER RETIRED IN THE I-TH AGE-
C                                  INTERVAL, XDI(I) GE.0, (F9.0).
C
C         THE ELEMENTS IN COMMON STORAGE ARE DEFINED AS FOLLOWS:
C             NINT   - NUMBER OF AGE-INTERVALS MINUS ONE
C             TINTV  - INITIAL AGE-INTERVAL VALUES
C             TMID   - AGE-INTERVAL MID-POINTS
C             TWID   - AGE-INTERVAL WIDTHS
C             XNI    - NUMBER EXPOSED IN EACH AGE-INTERVAL
C             XDI    - NUMBER RETIRED IN EACH AGE-INTERVAL
C             HAZD   - HAZARD RATE IN EACH AGE-INTERVAL
C             VHAZD  - VARIANCE OF HAZARD RATE IN EACH AGE-INTERVAL
C
C         DATA SET REFERENCE NUMBERS ARE IDENTIFIED AS FOLLOWS.
C             IN     - CARD READER.
C             LINE   - LINE PRINTER.
C
C
C
```

```
C          MAIN PROGRAM
C
      COMMON NINT,TINTV(100),TMID(100),TWID(100),XNI(100),XDI(100),
     .      HAZD(100),VHAZD(100),BUFF6(100,36)
C
      DIMENSION CLAMO(4,3),VLAMO(4,3),SELAMO(4,3),CLAM1(3,3,3),
     .          SELAM1(3,3),PARBUF(7,12),ALAB(4,8),NUM(4),FLNBUF(4,3),
     .          HAZBUF(100,12),P(100,12),SURBUF(100,12),PRDBUF(100,12),
     .          CSQ(4),PCSQ(4),NUMP1(3),SURCUM(100),DEN(100)
C
      EQUIVALENCE (BUFF6(1,1),HAZBUF(1,1)),(BUFF6(1,13),SURBUF(1,1)),
     .          (BUFF6(1,25),PRDBUF(1,1))
      REAL*8 ACNTNO,BCNTNO
      REAL*4 DESCR(20)
      DATA NUM/1,2,3,4/
      DATA NUMP1/2,3,4/
      DATA IN/5/, LINE/10/, IGDE1/8/, IGDE2/7/
      DATA ALAB/4HLAMB,4HDA-0,4H    ,4H    ,
     .          4HVAR(,4HLAMB,4HDA-0,4H)   ,
     .          4HST.E,4HRROR,4H(LAM,4H-0) ,
     .          4HLAMB,4HDA-1,4H    ,4H    ,
     .          4HVAR(,4HLAMB,4HDA-1,4H)   ,
     .          4HST.E,4HRROR,4H(LAM,4H-1) ,
     .          4HCOV(,4HLAM-,4HO,LA,4HM-1),
     .          4HLN-L,4HIKEL,4HIHOO,4HD   /
C
C          INPUT ANALYSIS DESCRIPTION CARD
C
10    READ(IN,240,END=530) ACNTNO,(DESCR(I),I=1,12),JMAX,XENT,METHOD
C
C          TEST FOR NEW PLANT ACCOUNT
C
      IF(ACNTNO.EQ.BCNTNO) GO TO 50
      BCNTNO = ACNTNO
C
C          INPUT INTERVAL DATA CARD
C
      READ(IN,250) (TINTV(I), XDI(I), I=1,JMAX)
C
C          COMPUTE SURVIVORS ENTERING EACH AGE-INTERVAL
C
      XNI(1) = XENT
      NINT = JMAX - 1
      DO 20 I=1,NINT
      XNI(I+1) = XNI(I) - XDI(I)
20    CONTINUE
C
C          DELETE INTERVALS AFTER ALL ARE RETIRED
C
      NINT = JMAX
30    IF(XNI(NINT).GT.0.0) GO TO 40
      NINT = NINT - 1
      IF(NINT.GT.0) GO TO 30
C
C          ERROR - NO. ENTERING 1ST INTERVAL LESS THAN OR EQUAL TO ZERO
```

```
C
      WRITE(LINE,260)
      GO TO 530
C
C          CALCULATE WIDTH AND MIDPOINT OF AGE-INTERVALS
C
40    IF(NINT.EQ.JMAX) NINT = JMAX - 1
      CALL WIDMID (TINTV,JMAX,TMID,TWID)
C
C          WRITE HEADING
C
50    WRITE(LINE,270) ACNTNO, (DESCR(I),I=1,12)
      GO TO (60,70,80), METHOD
60    WRITE(LINE,490)
      GO TO 90
70    WRITE(LINE,500)
      GO TO 90
80    WRITE(LINE,510)
90    CONTINUE
C
C          COMPUTE LIFE TABLE
C
      CALL LIFETB (SURCUM,DEN,FLNLSM,METHOD,LINE)
C
C          INITIALIZE BUFFERS
C
      CALL SETR (0.0,PARBUF,84)
      CALL SETR (1.0,FLNBUF,12)
      CALL SETR (1.0,BUFF6,3600)
C
C          COMPUTE LEAST SQUARES SOLUTION
C
      CALL LSQEST (CLAMO,VLAMO,SELAMO,CLAM1,SELAM1)
C
C          POSITION PARAMETER ESTIMATES IN BUFFER FOR OUTPUT
C
      DO 110 MM=1,4
      DO 100 MW=1,3
      MM1 = MM - 1
      J = ((MM-1)*3) + MW
      PARBUF(1,J) = CLAMO(MM,MW)
      PARBUF(2,J) = VLAMO(MM,MW)
      PARBUF(3,J) = SELAMO(MM,MW)
      IF(MM1.LE.0) GO TO 100
      PARBUF(4,J) = CLAM1(MM1,MW,1)
      PARBUF(5,J) = CLAM1(MM1,MW,2)
      PARBUF(6,J) = SELAM1(MM1,MW)
      PARBUF(7,J) = CLAM1(MM1,MW,3)
100   CONTINUE
110   CONTINUE
C
C          CHECK RANGE OF MODEL 1 AND MODEL 4 PARAMETERS FOR EACH WEIGHT
C
      DO 140 MW=1,3
      MM = 1
```

```
120   IF(CLAMO(MM,MW).GE.0.0) GO TO 130
      IF(MM.EQ.1) WRITE(LINE,430) MM,MW
      WRITE(LINE,440) MM,MW
      M = ((MM-1)*3) + MW
      IF(MM.EQ.1) CALL SETR (0.0,HAZBUF(1,M),NINT)
      CALL SETR (0.0,SURBUF(1,M),NINT+1)
      FLNBUF(MM,MW) = 0.0
130   IF(MM.EQ.4) GO TO 140
      MM = 4
      GO TO 120
140   CONTINUE
C
C          COMPUTE HAZARD FUNCTION
C
      CALL HAZFCN (CLAMO,CLAM1,FLNBUF,LINE)
C
C          COMPUTE SURVIVAL FUNCTION
C
      CALL SURFCN (P,LINE,FLNBUF,CLAM1,CLAMO)
C
C          COMPUTE LN-LIKELIHOOD FOR EACH MODEL
C
      CALL LNLIK(P,FLNBUF)
C
C          PRINT OUTPUT BUFFERS
C
      WRITE(LINE,300)
      WRITE(LINE,280)
      WRITE(LINE,290)
      WRITE(LINE,310) (NUM(I), I=1,4)
      WRITE(LINE,320) ((NUM(I), I=1,3), J=1,4)
C
C          OUTPUT PARAMETER ESTIMATES
      WRITE(LINE,330) ((ALAB(J,I), J=1,4), (PARBUF(I,J), J=1,12), I=1,3)
      WRITE(LINE,340) ((ALAB(J,I), J=1,4), (PARBUF(I,J), J=4,12), I=4,6)
      WRITE(LINE,350) (ALAB(J,7), J=1,4), (PARBUF(7,J), J=7,12)
C
C          OUTPUT LN-LIKELIHOOD VALUES
      WRITE(LINE,360) (ALAB(J,8), J=1,4), ((FLNBUF(I,J), J=1,3), I=1,4)
C
C          COMPUTE PROBABILITY DENSITY FUNCTION
C
      DO 160 I=1,NINT
      DO 150 J=1,12
      PROBUF(I,J) = HAZBUF(I,J)*PROBUF(I,J)
150   CONTINUE
160   CONTINUE
C
C          PRINT ESTIMATES OF HAZARD FUNCTION, SURVIVORSHIP FUNCTION,
C          AND PROBABILITY DENSITY FUNCTION
C
      DO 170 KK=1,3
      IF(KK.EQ.1) WRITE(LINE,370)
      IF(KK.EQ.2) WRITE(LINE,410)
      IF(KK.EQ.3) WRITE(LINE,420)
```

```
      IB = ((KK-1)*12) + 1
      IE = IB + 11
      WRITE(LINE,310) (NUM(I), I=1,4)
      WRITE(LINE,380) ((NUM(I), I=1,3),J=1,4)
      WRITE(LINE,390) (TINTV(I), (BUFF6(I,J), J=IB,IE), I=1,NINT)
      IF(KK.EQ.2) WRITE(LINE,390) TINTV(JMAX),(BUFF6(JMAX,J),J=1,IB,IE)
      IF(KK.NE.2) WRITE(LINE,400) TINTV(JMAX)
170   CONTINUE
C
C          DETERMINE IF DATA ARE EXPONENTIAL
C
C          CHOOSE LARGEST LN-LIKELIHOOD FOR MODEL 1
      BLNL = FLNBUF(1,1)
      NBLNL = 1
      DO 180 I=2,3
      IF(FLNBUF(1,I).LE.BLNL) GO TO 180
      BLNL = FLNBUF(1,I)
      NBLNL = I
180   CONTINUE
C
C          COMPUTE A CHI-SQ.  WITH 1 D.F. AND ASSOCIATED PROBABILITIES
C          FOR MODEL 1 VS 2, 1 VS 3, AND 1 VS 4 FOR THE WEIGHT SELECTED
C          ABOVE
      DO 190 I=2,4
      CSQ(I-1) = 0.0
      PCSQ(I-1) = 0.0
      IF(FLNBUF(I,NBLNL).EQ.0.0) GO TO 190
      CSQ(I-1) = 2.0*ABS(FLNBUF(1,NBLNL) - FLNBUF(I,NBLNL))
      PCSQ(I-1) = CHISQ(CSQ(I-1),1)
190   CONTINUE
C
C          CHI-SQ (.05) WITH 1 D.F. =3.8416. IF ALL CSQ-S ARE LT. 3.8416
C          CONSIDER DATA EXPONINTIAL - OTHERWISE SELECT MODEL WITH THE
C          LARGEST LN-LIKELIHOOD
      MM = 1
      CMM = 0.0
      MW = NBLNL
      DO 200 I=1,3
      IF(CSQ(I).EQ.0.0) GO TO 200
      IF(CSQ(I).GT.3.8416) GO TO 210
200   CONTINUE
      GO TO 230
210   MM =1
      MW = 1
      FM = FLNBUF(1,1)
      DO 220 LM =1,4
      DO 220 LW =1,3
      IF(FLNBUF(LM,LW).EQ.0.0) GO TO 220
      IF(FLNBUF(LM,LW).LE.FM) GO TO 220
      MM = LM
      MW = LW
      FM = FLNBUF(LM,LW)
220   CONTINUE
230   CONTINUE
C
```

```
C           DETERMINE GOODNESS OF FIT OF MODEL CHOSEN
C
C           COMPUTE CHI-SQ. WITH (S-1-K) D.F. WHERE S= NO. OF INTERVALS
C           AND K= THE NO. OF PARAMETERS IN THE MODEL, AND ASSOCIATED
C           PROBABILITIES FOR SAMPLE DATA VS. CHOSEN MODEL
      CSQ(4) = 2.0 * ABS(FLNLSM - FLNBUF(MM,MW))
      K = 1
      IF(MM.GT.1) K = 2
      IDF = NINT - K
      PCSQ(4) = CHISQ(CSQ(4), IDF)
C
C           PRINT RESULTS OF SELECTING BEST FIT
      WRITE(LINE,450) (NUMP1(I), NBLNL, CSQ(I), PCSQ(I), I=1,3)
      IF(MM.EQ.1) WRITE(LINE,460) NBLNL
      IF(MM.NE.1) WRITE(LINE,470) MM, NBLNL
      WRITE(LINE,480) MM, MW, FLNBUF(MM,MW), FLNLSM, MM, MW, CSQ(4),
     .                IDF, PCSQ(4)
C
      GO TO 10
C
C           FORMAT STATEMENTS
C
240   FORMAT(A8,2X,12A4,2X,I3,2X,F9.0,2X,I1)
250   FORMAT(10X,F9.2,2X,F9.0)
260   FORMAT(//,' ANALYSIS TERMINATED.  NO. OF UNITS ENTERING FIRST AGE-
     . INTERVAL IS LESS THAN OR EQUAL ZERO.')
270   FORMAT(1H1,//,37X,A8,2X,12A4)
280   FORMAT(////,10X,'MODEL 1 = EXPONENTIAL',/,10X,'MODEL 2 = LINEAR ',
     . 'HAZARD',/,10X,'MODEL 3 = GOMPERTZ',/,10X,'MODEL 4 = WEIBULL')
290   FORMAT(//,10X,'WEIGHT1(I) = 1.',/,10X,'WEIGHT2(I) = 1. / V',/,10X,
     . 'WEIGHT3(I) = N(I) * H(I)')
300   FORMAT(1H1,////,49X,'ESTIMATES OF PARAMETERS')
310   FORMAT((/,28X,4('MODEL ',I1,19X)))
320   FORMAT(20X,4(3('  WT ',I1,2X),2X))
330   FORMAT(2(1X,4A4,1X,4(3F8.4,2X),/),1X,4A4,1X,4(3F8.4,2X))
340   FORMAT(2(1X,4A4,27X,3(3F8.4,2X),/),1X,4A4,27X,3(3F8.4,2X))
350   FORMAT(1X,4A4,53X,2(3F8.4,2X))
360   FORMAT(1X,4A4,1X,4(3F8.2,2X))
370   FORMAT(////,47X,'ESTIMATES OF HAZARD FUNCTION')
380   FORMAT(1X,'INTERVAL START',5X,4(3('  WT ',I1,2X),2X))
390   FORMAT(4X,F7.2,7X,3F8.4,2X,3F8.4,2X,3F8.4,2X,3F8.4)
400   FORMAT(4X,F7.2,12X,4(3('***',6X),2X))
410   FORMAT(1H1,////,44X,'ESTIMATES OF SURVIVORSHIP FUNCTION')
420   FORMAT(////,41X,'ESTIMATES OF PROBABILITY DENSITY FUNCTION')
430   FORMAT(//,' MODEL ',I1,', WEIGHT ',I1,' IS INAPPROPRIATE SINCE',
     . ' THE ESTIMATE OF THE HAZARD FUNCTION IS NEGATIVE.')
440   FORMAT(//,' MODEL ',I1,', WEIGHT ',I1,' IS INAPPROPRIATE SINCE',
     . ' THE ESTIMATE OF THE SURVIVORSHIP FUNCTION IS GREATOR THAN ',
     . '1.0.')
450   FORMAT(1H1,/////,' TEST OF WHETHER DATA ARE EXPONENTIAL',///,
     . 5X,'MODELS    WT.     CHI-SQ   D.F.    P',3(/,5X,'1 VS ',I1,4X,I1,3X,
     . F8.3,3X,'I',3X,F4.2))
460   FORMAT(///,5X,'TEST INDICATES DATA CAN BE FITTED BY MODEL 1, ',
     . 'WEIGHT ',I1)
470   FORMAT(///,5X,'TEST INDICATES DATA CAN BEST BE FITTED BY MODEL ',
```

```
     . I1,', WEIGHT ',I1)
480   FORMAT(/////,' TEST OF GOODNESS OF FIT OF CHOSEN MODEL',///,23X,
     . 'MODEL ',I1,', WT. ',I1,3X,'SAMPLE DATA',/,5X,'LN-LIKELIHOOD',
     . 2(7X,F8.2),///,42X,'CHI-SQ    D.F.      P',/,5X,'MODEL ',I1,
     . ' WT. ',I1,' VS. SAMPLE DATA',7X,F8.3,3X,I2,4X,F4.2)
490   FORMAT(/,50X,'CONDITIONAL PROPORTION RETIRED')
500   FORMAT(/,56X,'ACTUARIAL ESTIMATE')
510   FORMAT(/,52X,'MAXIMUM LIKELIHOOD ESTIMATE')
530   STOP
C
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE WIDMID (TM,INT,XMID,H)
      DIMENSION TM(1),XMID(1),H(1)
C
C          SUBROUTINE TO CALCULATE WIDTHS AND MIDPOINTS OF GIVEN
C          AGE-INTERVALS
C
C          THE INPUT PARAMETERS ARE DEFINED AS FOLLOWS
C             TM     - INITIAL AGE-INTERVAL VALUES
C             INT    - NUMBER OF AGE INTERVALS
C
      IM1 = INT - 1
      DO 10 I=1,IM1
      IP1 = I + 1
      H(I) = TM(IP1) - TM(I)
      XMID(I) = TM(I) + H(I)/2.0
10    CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE LIFETB (SURCUM,DEN,FLNLSM,METHOD,LINE)
C
C          SUBROUTINE TO COMPUTE LIFE TABLE DATA
C
C          THE INPUT PARAMETERS ARE DEFINED AS FOLLOWS,
C             XNI    - NUMBER ENTERING AGE-INTERVAL
C
C          THE OUTPUT PARAMETERS ARE DEFINED AS FOLLOWS,
C             SURCUM - CUMULATIVE PROPORTION SURVIVING (I.E., THE
C                      SURVIVORSHIP FUNCTION FOR THE SAMPLE DATA)
C             DEN    - PROBABILITY DENSITY FUNCTION FOR THE SAMPLE DATA
C
      DIMENSION DYPN(100),SURPN(100),SURCUM(100),DEN(100),SCSUR(100),
     .          SDEN(100),SHAZ(100),ELIF(100),PLS(100),SELIF(100),PL(2)
C
      COMMON NINT,TINTV(100),THID(100),TWID(100),XNI(100),XDI(100),
     .       HAZD(100),VHAZD(100),BUFF6(100,36)
C
      DATA  PL/1H ,1H+/
C
```

```
C           INITIALIZATION
C
      INTP1 = NINT + 1
      SURCUM(1) = 1.0
C
C           COMPUTE NO. EXPOSED AND PROPORTION RETIRED
C
      DO 80 I=1,INTP1
C
C           IN THE FINAL INTERVAL XNI IS ALLOWED TO BE ZERO
C
      IF(I.NE.INTP1 .OR. XNI(I).NE.0.0) GO TO 10
      DYPN(I) = 0.
      GO TO 20
10    DYPN(I) = XDI(I)/XNI(I)
C
C           CORRECT FOR DYPN = 0 OR DYPN = 1
C
      IF(DYPN(I).EQ.1.0) DYPN(I) = (XNI(I) - 0.5)/XNI(I)
      IF(DYPN(I).EQ.0.0) DYPN(I) = 0.5/XNI(I)
C
C           COMPUTE PROPORTION SURVIVING AND CUMULATIVE PROPORTION
C           SURVIVING
C
20    SURPN(I) = 1.0 - DYPN(I)
30    IF(I.EQ.1) GO TO 40
      IM1 = I - 1
      SURCUM(I) = SURPN(IM1) * SURCUM(IM1)
C
C           COMPUTE PROBABILITY DENSITY - UNDEFINED IN THE LAST INTERVAL
C
      DEN(IM1) = (SURCUM(IM1) - SURCUM(I))/TWID(IM1)
C
C           COMPUTE HAZARD AND VARIANCE OF HAZARD
C
40    IF(I.EQ.INTP1) GO TO 80
C
C           BRANCH TO SELECTED METHOD FOR ESTIMATING HAZARD RATE
C
      GO TO (50,60,70), METHOD
C
C           ACTUARIAL METHOD ONE
C
50    HAZD(I) = DYPN(I)/TWID(I)
      VHAZD(I) = DYPN(I)*SURPN(I)/XNI(I)/TWID(I)**2
      SHAZ(I) = SQRT(VHAZD(I))
      GO TO 80
C
C           ACTURIAL METHOD TWO
C
60    HAZD(I) = (2.0 * DYPN(I))/(TWID(I) * (1.0 + SURPN(I)))
      VHAZD(I) = ((HAZD(I) ** 2)/(XNI(I) * (1.0 - SURPN(I)))) *
     *            (1.0 - ((HAZD(I) * TWID(I))/2.0) ** 2)
      SHAZ(I) = SQRT(VHAZD(I))
      GO TO 80
```

```
C
C          MAXIMUM LIKELIHOOD METHOD
C
70    HAZD(I) = -ALOG(SURPN(I))/TWID(I)
      VHAZD(I) = DYPN(I)/((TWID(I) ** 2) * XNI(I) * SURPN(I))
      SHAZ(I) = SQRT(VHAZD(I))
      GO TO 80
C
80    CONTINUE
C
C          STANDARD ERROR COMPUTATIONS
C
      DO 110 I=1,INTP1
      SUM1 = 0.0
      IF(I.EQ.1) GO TO 100
      IM1 = I - 1
      DO 90 IM=1,IM1
      SUM1 = SUM1 + (DYPN(IM))/(XNI(IM) * SURPN(IM))
90    CONTINUE
100   VCSUR = (SURCUM(I) ** 2) * SUM1
      SCSUR(I) = SQRT(VCSUR)
      IF(I.EQ.INTP1) GO TO 110
      Q1 = ((SURCUM(I) * DYPN(I)) ** 2)/(TWID(I) ** 2)
      Q2 = SUM1 + (SURPN(I)/(XNI(I) * DYPN(I)))
      VDEN = Q1 * Q2
      SDEN(I) = SQRT(VDEN)
110   CONTINUE
C
C          MEDIAN LIFE EXPECTENCY COMPUTATIONS
C
      DO 150 I=1,NINT
      PSRCH = 0.5 * SURCUM(I)
      DO 120 IP=1,INTP1
      IPM1 = IP - 1
      IF(PSRCH.LT.SURCUM(INTP1)) GO TO 140
      IF(PSRCH.GT.SURCUM(IP) .AND. PSRCH.LE.SURCUM(IPM1)) GO TO 130
120   CONTINUE
130   ELIF(I) = (TINTV(IPM1) - TINTV(I)) + (TWID(IPM1) * ((SURCUM(IPM1)
     .           - PSRCH)/(SURCUM(IPM1) - SURCUM(IP))))
      PLS(I) = PL(1)
      SELIF(I) = SQRT((SURCUM(I) ** 2)/(4.0 * XNI(I) * DEN(IPM1)
     .           ** 2))
      GO TO 150
140   ELIF(I) = TINTV(INTP1) - TINTV(I)
      PLS(I) = PL(2)
      SELIF(I) = 0.0
150   CONTINUE
C
C          CALL 'LNLIKS' TO CALCULATE LN-LIKELIHOOD FOR SAMPLE DATA
C
      CALL LNLIKS (XDI,XNI,SURCUM,NINT,FLNLSM)
C
C         PRINT LIFE TABLE
C
160   WRITE(LINE,170)
```

```
      WRITE(LINE,180) (TINTV(I), TMID(I), TWID(I), XNI(I), XDI(I),
     .                 DYPN(I), SURPN(I), SURCUM(I), DEN(I),
     .                 HAZD(I), SCSUR(I), SDEN(I), SHAZ(I), ELIF(I),
     .                 PLS(I), SELIF(I), I=1,NINT)
      WRITE(LINE,190) TINTV(INTP1), XNI(INTP1), XDI(INTP1),
     .                DYPN(INTP1), SURPN(INTP1), SURCUM(INTP1),
     .                SCSUR(INTP1)
      WRITE(LINE,200)
      WRITE(LINE,210) FLNLSM
      RETURN
C
C          FORMAT STATEMENTS
C
170   FORMAT(////,58X,'LIFE TABLE DATA',//,9X,' INT    MID    INT',2('
     .    NO.'),'    PROPN   PROPN    CUM     PROB   HAZD',3('  ST ER'),'
     .    MED',5X,'ST ER',/,8X,' START  POINT  WIDTH    ENTER   RETIRE
     .RETIRE    SURV   PROPN   DENS   RATE   CUM    PROB   HAZD    LIFE
     .    LIFE',/,66X,'SURV',17X,'SURV   DENS    RATE   EXPECT   EXPECT')
180   FORMAT (7X,F6.1,F8.2,F7.2,2F9.0,3F8.4,5F7.4,F9.4,A1,F8.4)
190   FORMAT (7X,F6.1,6X,'**',5X,'**',2F9.0,3F8.4,2(5X,'**'),F7.4,2(5X,'
     .**'),8X,'*',7X,'*')
200   FORMAT (////,7X,' *        INDICATES NO MEDIAN LIFE EXPECTANCY CAN
     .BE CALCULATED FOR THIS ENTRY.'/,7X,' **       CALCULATIONS INVOLVIN
     .G INTERVAL WIDTH FOR LAST INTERVAL HAVE NO MEANING.')
210   FORMAT (////,7X' LN-LIKELIHOOD FOR SAMPLE DATA = ',F12.2)
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE SETR (FINDX,FIN,N)
C
C          SET N ELEMENTS IN FLOATING POINT ARRAY FIN
C          EQUAL TO THE FLOATING POINT CONSTANT FINDX.
C
      DIMENSION FIN(1)
      DO 10 I=1,N
      FIN(I) = FINDX
10    CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE LSQEST (CLAMO,VLAMO,SELAMO,CLAM1,SELAM1)
C
C          LEAST SQUARE ESTIMATES OF HAZARD RATES
C
C          THE INPUT PARAMETERS ARE DEFINED AS FOLLOWS.
C              TMID   - AGE-INTERVAL MIDPOINTS.
C              XNI    - NUMBER ENTERING EACH AGE-INTERVAL.
C              HAZD   - HAZARD RATE IN EACH AGE-INTERVAL.
C              VHAZD  - VARIANCE OF HAZARD RATE IN EACH AGE-INTERVAL
C              NINT   - NUMBER OF AGE-INTERVALS.
C
C           THE OUTPUT PARAMETERS ARE DEFINED AS FOLLOWS.
```

```
C              CLAMO  - LAMBDA-O VALUES.
C              VLAMO  - VARIANCE OF LAMBDA-O.
C              SELAMO - STANDARD ERROR OF LAMBDA-O
C              CLAM1  - LAMBDA-1 AND VARIANCE OF LAMBDA-1 VALUES.
C              SELAM1 - STANDARD ERROR OF LAMBDA-1.
C
      COMMON NINT,TINTV(100),TMID(100),TWID(100),XNI(100),XDI(100),
     .       HAZD(100),VHAZD(100),BUFF6(100,36)
C
      DIMENSION V(100,100),W(100,100),TA(100,1),Y(100,1),TB(100,2),
     .       TAT(1,100),TBT(2,100),VHAZ(100,4),TEMP(1,100),SELAMO(4,3),
     .       TEA(1,1),PL(1,100),TEMPB(2,100),T2(2,2),PLB(2,100),
     .       A234(2,1),CLAMO(4,3),CLAM1(3,3,3),VLAMO(4,3),SELAM1(3,3)
C
      DATA LINE/10/
C
C
C
C          EXPAND VARIANCE OF HAZARD FOR 4 MODELS.
C
      DO 10 I=1,NINT
      DO 10 J=1,4
      VHAZ(I,J) = VHAZD(I)
10    CONTINUE
C          INITIALIZATION
C
      DO 20 I=1,4
      DO 20 J=1,3
      CLAMO(I,J) = 0.0
      VLAMO(I,J) = 0.0
20    CONTINUE
      DO 30 I=1,3
      DO 30 J=1,3
      DO 30 K=1,3
      CLAM1(I,J,K) = 0.0
30    CONTINUE
C
C          MM IS MODEL, MW IS MODEL WEIGHT. INITIALIZE AND INCREMENT.
C
      MM = 0
40    MM = MM + 1
      IF(MM.GT.4) GO TO 360
      MW = 0
50    MW = MW + 1
      IF(MW.GT.3) GO TO 40
C
C          FILL Y, TR, W, AND V MATRICES WITH HAZARD RATE, MIDPOINT OF
C          AGE-INTERVAL, MODEL WEIGHT, AND VARIANCE OF HAZARD.
C
      CALL SETR (0.0,V,10000)
      CALL SETR (0.0,W,10000)
      DO 110 I=1,NINT
      GO TO (60,70,80,90), MM
60    TA(I,1) = 1.0
      Y(I,1) = HAZD(I)
```

```
      V(I,I) = VHAZ(I,MM)
      GO TO 100
70    TB(I,1) = 1.0
      TB(I,2) = TMID(I)
      Y(I,1) = HAZD(I)
      V(I,I) = VHAZ(I,MM)
      GO TO 100
80    TB(I,1) = 1.0
      TB(I,2) = TMID(I)
      Y(I,1) = ALOG(HAZD(I))
      V(I,I) = VHAZ(I,MM)/(HAZD(I)**2)
      GO TO 100
90    TB(I,1) = 1.0
      TB(I,2) = ALOG(TMID(I))
      Y(I,1) = ALOG(HAZD(I))
      V(I,I) = VHAZ(I,MM)/(HAZD(I)**2)
100   W(I,I) = 1.0
      IF(MW.EQ.2) W(I,I) = 1.0/V(I,I)
      IF(MW.EQ.3) W(I,I) = XNI(I)*TWID(I)
110   CONTINUE
C
C          FIND L(TRANS) = ((T(TRANS) * W * T) ** -1) * T(TRANS) * W
C
C           FIND T(TRANS)
      NR = NINT
      NC = 1
      IF(MM.GT.1) NC = 2
      IF(NC.EQ.2) GO TO 130
      DO 120 I=1,NINT
120   TAT(1,I) = TA(I,1)
      GO TO 150
130   DO 140 I=1,NINT
      DO 140 J=1,2
140   TBT(J,I) = TB(I,J)
150   CONTINUE
      IF(NC.EQ.2) GO TO 200
C
C          MULTIPLY TAT BY W TO TEMP GIVING T(TRANS) * W
      DO 160 I=1,NC
      DO 160 J=1,NR
      TEMP(I,J) = 0.0
      DO 160 K=1,NR
160   TEMP(I,J) = TEMP(I,J) + TAT(I,K)*W(K,J)
C
C          MULTIPLY TEMP BY TA TO TEA GIVING T(TRANS) * W * T
      TEA(1,1) = 0.0
      DO 170 K=1,NINT
170   TEA(1,1) = TEA(1,1) + TEMP(1,K)*TA(K,1)
C
C          FIND INVERSE OF A 1 BY 1 MATRIX GIVING
C          (T(TRANS) * W * T) ** -1
      TEA(1,1) = 1.0/TEA(1,1)
C
C          MULTIPLY TEA BY TAT TO TEMP
      DO 180 J=1,NINT
```

```
180   TEMP(1,J) = TEA(1,1)*TAT(1,J)
C
C          MULTIPLY TEMP BY W TO PL
      DO 190 J=1,NINT
      PL(1,J) = 0.0
      DO 190 K=1,NINT
190   PL(1,J) = PL(1,J) + TEMP(1,K)*W(K,J)
      GO TO 250
200   CONTINUE
C
C          MULTIPLY TBT BY W TO TEMPB
      DO 210 I=1,NC
      DO 210 J=1,NR
      TEMPB(I,J) = 0.0
      DO 210 K=1,NR
210   TEMPB(I,J) = TEMPB(I,J) + TBT(I,K)*W(K,J)
C
C          MULTIPLY TEMPB BY TB TO T2
      DO 220 I=1,2
      DO 220 J=1,2
      T2(I,J) = 0.0
      DO 220 K=1,NINT
220   T2(I,J) = T2(I,J) + TEMPB(I,K)*TB(K,J)
C
C          FIND THE INVERSE OF T2, A 2 BY 2 MATRIX = (D/DET   -B/DET)
C                                                   (C/DET    A/DET)
      DETE = T2(1,1)*T2(2,2) - T2(1,2)*T2(2,1)
      AL1 = T2(2,2)/DETE
      AL7 = (-T2(1,2))/DETE
      AL3 = (-T2(2,1))/DETE
      AL4 = T2(1,1)/DETE
      T2(1,1) = AL1
      T2(1,2) = AL7
      T2(2,1) = AL3
      T2(2,2) = AL4
C
C          MULTIPLY T2 BY TBT TO TEMPB GIVING
C          ((T(TRANS) * W * T) ** -1) * T(TRANS)
      DO 230 I=1,2
      DO 230 J=1,NINT
      TEMPB(I,J) = 0.0
      DO 230 K=1,2
230   TEMPB(I,J) = TEMPB(I,J) + T2(I,K)*TBT(K,J)
C
C          MULTIPLY TEMPB BY W TO PLB GIVING L(TRANS)
      DO 240 I=1,2
      DO 240 J=1,NINT
      PLB(I,J) = 0.0
      DO 240 K=1,NINT
240   PLB(I,J) = PLB(I,J) + TEMPB(I,K)*W(K,J)
250   IF(MM.NE.1) GO TO 290
C
C          FIND ESTIMATE - LAMBDA AND VARIANCE OF LAMBDA, MODEL 1
      ESTI = 0.0
      DO 260 I=1,NINT
```

```
260   ESTI = ESTI + PL(1,I)*Y(I,1)
C
C           MULTIPLY PL BY V TO TEMP
      DO 270 J=1,NINT
      TEMP(1,J) = 0.0
      DO 270 K=1,NINT
270   TEMP(1,J) = TEMP(1,J) + PL(1,K)*V(K,J)
      ESTV = 0.0
      DO 280 I=1,NINT
280   ESTV = ESTV + TEMP(1,I)*PL(1,I)
      CLAMO(1,MW) = CLAMO(1,MW) + ESTI
      VLAMO(1,MW) = VLAMO(1,MW) + ESTV
      GO TO 50
290   CONTINUE
C
C           MULTIPLY PLB BY Y TO A234 GIVING
C           LAMBDA = L(TRANS) * Y FOR MODELS 2, 3, AND 4
      DO 300 I=1,2
      A234(I,1) = 0.0
      DO 300 K=1,NINT
300   A234(I,1) = A234(I,1) + PLB(I,K)*Y(K,1)
C
C           MULTIPLY PLB BY V TO TEMPB GIVING VARIANCE OF
C           LAMBDA = L(TRANS) * V * L FOR MODELS 2, 3, AND 4
      DO 310 I=1,2
      DO 310 J=1,NINT
      TEMPB(I,J) = 0.0
      DO 310 K=1,NINT
310   TEMPB(I,J) = TEMPB(I,J) + PLB(I,K)*V(K,J)
C
C           TRANSPOSE L
      DO 320 I=1,NINT
      DO 320 J=1,2
320   TB(I,J) = PLB(J,I)
C
C           MULTIPLY TEMPA BY TB TO T2
      DO 330 I=1,2
      DO 330 J=1,2
      T2(I,J) = 0.0
      DO 330 K=1,NINT
330   T2(I,J) = T2(I,J) + TEMPB(I,K)*TB(K,J)
      IF(MM.EQ.4) GO TO 340
      GO TO 350
340   ESTI = A234(2,1) + 1.0
      ESTV = EXP(A234(1,1))/ESTI
      A234(1,1) = ESTV
      A234(2,1) = ESTI
      ESTV = (A234(1,1)**2*T2(1,1)) + ((A234(1,1)**2/A234(2,1)**2)
     1*T2(2,2)) - (((2.0*A234(1,1)**2)/A234(2,1))*T2(1,2))
      ESTI = (A234(1,1)*T2(1,2)) - ((A234(1,1)*T2(2,2))/A234(2,1))
      T2(1,1) = ESTV
      T2(1,2) = ESTI
      T2(2,1) = ESTI
C
C           STORE FINAL RESULTS
```

```
350   CLAM0(MM,MW) = A234(1,1)
      VLAM0(MM,MW) = T2(1,1)
      CLAM1(MM-1,MW,1) = A234(2,1)
      CLAM1(MM-1,MW,2) = T2(2,2)
      CLAM1(MM-1,MW,3) = T2(1,2)
      GO TO 50
360   CONTINUE
C
C          COMPUTE STANDARD ERRORS
      DO 370 I=1,4
      DO 370 J=1,3
      SELAM0(I,J) = SQRT(VLAM0(I,J))
      IF(I.EQ.1) GO TO 370
      SELAM1(I-1,J) = SQRT(CLAM1(I-1,J,2))
370   CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE STOR (I,MW,FACT,BUFF6)
C
C          SUBROUTINE TO STORE SURVIVAL AND HAZARD FUNCTION VALUES FOR
C          EACH OF 4 MODELS WITH A GIVEN MODEL WEIGHT IN THE PROPER
C          LOCATION OF A GIVEN BUFFER.
C
C          THE INPUT PARAMETERS ARE DEFINED AS FOLLOWS.
C              I     - DEFINES THE STORAGE ROW IN BUFF6 OR AGE-INTERVAL
C              MW    - MODEL WEIGHT
C              FACT  - VALUES TO BE STORED
C
C          THE OUTPUT PARAMETER IS DEFINED AS FOLLOWS.
C              BUFF6 - STORAGE BUFFER
C
      DIMENSION BUFF6(100,12),FACT(4)
C
C          STORE VALUES
      DO 10 MM=1,4
      J = MW + ((MM-1)*3)
      IF(BUFF6(I,J).NE.0.0) BUFF6(I,J) = FACT(MM)
10    CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE HAZFCN (CLAM0,CLAM1,FLNBUF,LINE)
C
C          SUBROUTINE TO COMPUTE THE HAZARD FUNCTION FOR EACH
C          AGE-INTERVAL.
C
      COMMON NINT,TINTV(100),TMID(100),TWID(100),XNI(100),XDI(100),
     *       HAZD(100),VHAZD(100),BUFF6(100,36)
C
      DIMENSION HAZBUF(100,12),CLAM0(4,3),CLAM1(3,3,3),FLNBUF(4,3),
```

```
     .              HAZF(4)
C
      EQUIVALENCE (BUFF6(1,1), HAZBUF(1,1))
C
C           COMPUTATION FOR 4 MODELS AND NINT AGE-INTERVALS.
C
      DO 30 MW=1,3
      IFLAG = 1
      DO 20 I=1,NINT
C
C           MODEL 1 - EXPONENTIAL
      HAZF(1) = CLAMO(1,MW)
C
C           MODEL 2 - LINEAR
      HAZF(2) = CLAMO(2,MW) + (CLAM1(1,MW,1)*TMID(I))
C
C           CHECK RANGE OF HAZARD FUNCTION
      IF(HAZF(2).GE.0.0) GO TO 10
      HAZF(2) = 0.0
      FLNBUF(2,MW) = 0.0
      IF(IFLAG.NE.0) GO TO 10
      IFLAG = 0
      MM=2
      WRITE(LINE,40) MM,MW
C
C           MODEL 3 - GOMPERTZ
10    FF = CLAMO(3,MW) + (CLAM1(2,MW,1)*TMID(I))
      HAZF(3) = EXP(FF)
C
C         MODEL 4 WEIBULL
      F1 = ALOG(CLAMO(4,MW)*CLAM1(3,MW,1))
      F2 = (CLAM1(3,MW,1) - 1.0)*ALOG(TMID(I))
      HAZF(4) = EXP(F1 + F2)
C
C           STORE HAZARD FUNCTION VALUES
.     CALL STOR (I,MW,HAZF,HAZBUF)
20    CONTINUE
30    CONTINUE
      RETURN
40    FORMAT(//,' MODEL ',I1,', WEIGHT ',I1,' IS INAPPROPRIATE SINCE',
     . ' THE ESTIMATE OF THE HAZARD FUNCTION IS NEGATIVE.')
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE SURFCN(P,LINE,FLNBUF,CLAM1,CLAMO)
C
C           SUBROUTINE TO COMPUTE THE SURVIVORSHIP FUNCTION IN EACH AGE-
C           INTERVAL AND THE PROPORTION SURVIVING EACH AGE-INTERVAL (TO
C           BE USED IN LN-LIKELHOOD CALCULATIONS) FOR EACH OF 4 MODELS
C           AND 3 WEIGHTS
C
C         THE OUTPUT PARAMETER IS DEFINED AS FOLLOWS:
C            P      - PROPORTION SURVIVING
C
```

```
      DIMENSION P(100,12),SURF(4),SURBUF(100,12),PRDBUF(100,12),
     .          FLNBUF(4,3),CLAMO(4,3),CLAM1(3,3,3)
C
      COMMON NINT,TINTV(100),TMID(100),TWID(100),XNI(100),XDI(100),
     .       HAZD(100),VHAZD(100),BUFF6(100,36)
C
      EQUIVALENCE (BUFF6(1,13),SURBUF(1,1)), (BUFF6(1,25),PRDBUF(1,1))
C
C         SET FIRST INTERVAL
      DO 10 I=1,12
      SURBUF(1,I) = 1.0
      PRDBUF(1,I) = 1.0
10    CONTINUE
C
C          COMPUTATIONS FOR 4 MODELS AND JMAX AGE-INTERVALS
      J1 = NINT + 1
      DO 60 MW=1,3
      IFLG2 = 0
      IFLG3 = 0
      DO 50 I=2,J1
C
C          COMPUTATIONS FOR THE SURVIVAL FUNCTION USING THE LOWER TIME
C          BOUNDRY OF THE AGE-INTERVAL AND FOR THE PROBABILITY DENSITY
C          FUNCTION USING TMID, THE INTERVAL MIDPOINT
      DO 40 KK=1,2
      IF(I.EQ.J1.AND.KK.EQ.2) GO TO 40
      IF(KK.EQ.1) TT = TINTV(I)
      IF(KK.EQ.2) TT = TMID(I)
C
C          MODEL 1 - EXPONENTIAL
      FF = -CLAMO(1,MW)*TT
      SURF(1) = EXP(FF)
C
C          MODEL 2 - LINEAR
      FF = -((CLAMO(2,MW)*TT) + ((CLAM1(1,MW,1)*(TT**2))/2))
      SURF(2) = EXP(FF)
      IF(SURF(2).LE.1.0) GO TO 20
      SURF(2) = 0.0
      IF(IFLG2.NE.0) GO TO 20
      IFLG2 = 1
      FLNBUF(2,MW) = 0.0
      MM=2
      WRITE(LINE,100) MM,MW
C
C          MODEL 3 - GOMPERTZ
20    FF1 = -(EXP(CLAMO(3,MW)))/CLAM1(2,MW,1)
      FF2 = EXP(CLAM1(2,MW,1)*TT) - 1
      SURF(3) = EXP(FF1*FF2)
      IF(SURF(3).LE.1.0) GO TO 30
      SURF(3) = 0.0
      IF(IFLG3.NE.0) GO TO 30
      IFLG3 = 1
      FLNBUF(3,MW) = 0.0
      MM=3
      WRITE(LINE,100) MM,MW
```

```
C
C          MODEL 4 WEIBULL
30    FF = -(CLAMO(4,MW)*(TT**CLAM1(3,MW,1)))
      SURF(4) = EXP(FF)
C
C          STORE IN SURBUF
      IF(KK.EQ.1) CALL STOR (I,MW,SURF,SURBUF)
      IF(KK.EQ.2) CALL STOR (I,MW,SURF,PRDBUF)
40    CONTINUE
50    CONTINUE
60    CONTINUE
C
C          COMPUTE P - THE PROPORTION SURVIVING
      DO 80 J=1,12
      DO 70 I=1,NINT
      P(I,J) = SURBUF(I+1,J)/SURBUF(I,J)
      TEST = 1.0 - (1.0/(2.0*XNI(I)))
      IF(P(I,J).GT.TEST) P(I,J) = TEST
      IF(P(I,J).GT.0.0) GO TO 70
C
C          ERROR CONDITION - PROPORTION SURVIVING LESS THAN OR EQUAL 0
      SURBUF(I+1,J) = 0.0
      MM = ((J-1)/3) + 1
      MW = J - ((MM-1)*3)
      FLNBUF(MM,MW) = 0.0
      WRITE(LINE,90) MM,MW
70    CONTINUE
80    CONTINUE
      RETURN
C
C         FORMAT STATEMENTS
90    FORMAT(//,' MODEL ',I1,', WEIGHT ',I1,', IS INAPPROPRIATE SINCE ',
     . 'THE COMPUTED CUMULATIVE PROPORTION SURVIVING IS NEGATIVE OR ',
     . 'ZERO.')
100   FORMAT(//,' MODEL ',I1,', WEIGHT ',I1,', IS INAPPROPRIATE SINCE',
     . ' THE ESTIMATE OF THE SURVIVORSHIP FUNCTION IS GREATER THAN',
     . '1.0.')
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE LNLIK (P,FLNBUF)
C
C          SUBROUTINE TO COMPUTE THE LN-LIKELIHOOD FOR EACH MODEL,
C          ACCORDING TO THE FOLLOWING FORMULA.
C
C                              N
C             FLNBUF(J,K) = SUM (XDI(I)*ALOG(1.0 - P(I,JK)) +
C                             I=1
C
C                              N
C                             SUM ((XNI(I) - XDI(I))*ALOG(P(I,IJ))
C                             I=1
C
C             WHERE  J=1,...,4;  K=1,...,3;  AND JK=((J-1)*3)+K
```

```
C
C           THE INPUT PARAMETER IS DEFINED AS FOLLOWS.
C              P      - ARRAY CONTAINING THE PROPORTION SURVIVING IN
C                       EACH AGE-INTERVAL COMPUTED FOR EACH OF THE 4
C                       MODELS
C
      DIMENSION P(100,12),FLNBUF(4,3)
C
      COMMON NINT,TINTV(100),TMID(100),TWID(100),XNI(100),XDI(100),
     .       HAZD(100),VHAZD(100),BUFF6(100,36)
C
      DO 20 J=1,12
      SUM1 = 0.0
      SUM2 = 0.0
      MM = ((J-1)/3) + 1
      MW = J - ((MM-1)*3)
      IF(FLNBUF(MM,MW).EQ.0.0) GO TO 20
      DO 10 I=1,NINT
      D = ALOG(1.0 - P(I,J)) * XDI(I)
      SUM1 = SUM1 + D
      S = ALOG(P(I,J))*(XNI(I) - XDI(I))
      SUM2 = SUM2 + S
10    CONTINUE
      FLNBUF(MM,MW) = SUM1 + SUM2
20    CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      FUNCTION CHISQ(XSQ,IDF)
C
C          FUNCTION ROUTINE TO COMPUTE THE CHISQ
C
      PI = 3.1415927
      X = SQRT(XSQ)
      S2PI = SQRT(2.0 * PI)
      IF(XSQ.LT.-180..OR.XSQ.GT.174) XSQ=0.
      Z = (1.0/S2PI) * EXP(-XSQ/2.0)
C
C          TEST IDF - EVEN OF ODD
      ITRY = IDF/2
      IF((ITRY * 2) - IDF) 50,10,50
C
C          CASE 1 - IDF EVEN
10    SUM = 0.0
      LOOP = (IDF -2)/2
      IF(LOOP.EQ.0) GO TO 40
      DO 30 L=1,LOOP
      DIV = 1.0
      DO 20 I=1,L
      FI = I
      DIV = DIV * (2.0 * FI)
20    CONTINUE
      EX = 2 * L
```

```
      SUM = SUM + (X ** EX)/DIV
30    CONTINUE
40    CHISQ = S2PI * 7 * (1.0 + SUM)
      RETURN
C
C          CASE 2 - IDF ODD
50    A1 = .43618
      A2 = -.12017
      A3 = .93730
      PP = .33267
      T = 1.0/(1.0 + (PP * X))
      QX = Z * ((A1 * T) + (A2 * (T ** 2)) + (A3 * (T ** 3)))
      SUM = 0.0
      LOOP = (IDF - 1)/2
      IF(LOOP.EQ.0) GO TO 80
      DO 70 L=1,LOOP
      DIV = 1.0
      DO 60 I=1,L
      FI = I
      DIV = DIV * ((2.0 * FI) - 1.0)
60    CONTINUE
      EX = (2 * L) - 1
      SUM = SUM + (X ** EX)/DIV
70    CONTINUE
80    CHISQ = (2.0 * QX) + (2.0 * Z * SUM)
      RETURN
      END
C
C-----------------------------------------------------------------------
C
      SUBROUTINE LNLIKS (XDI,XNI,SURCUM,NINT,FLNLSM)
C
C          SUBROUTINE TO COMPUTE LN-LIKELIHOOD FOR SAMPLE DATA.
C
      DIMENSION XDI(100),XNI(100),SURCUM(100)
C
      SUM1 = 0.0
      SUM2 = 0.0
      DO 10 I=I,NINT
      P = SURCUM(I+1)/SURCUM(I)
      SUM1 = SUM1 + ALOG(1.0-P) * XDI(I)
      SUM2 = SUM1 + ALOG(P) * (XNI(I)-XDI(I))
10    CONTINUE
      FLNLSM = SUM1 + SUM2
      RETURN
      END
```

APPENDIX C: EXAMPLE OF OBSERVED LIFE TABLE AND WEIGHTED LEAST SQUARES ESTIMATE OF PARAMETERS

V- 1 WEIBULL -- LAMBDA-0 = 0.08 LAMBDA-1 = 1.5

CONDITIONAL PROPORTICN RETIRED

LIFE TABLE DATA

INT START	MID POINT	INT WIDTH	NO. ENTER	NO. RETIRE	PROPN RETIRE	PROPN SURV	CUM PROPN SURV	PROB DENS	HAZD RATE	ST ER CUM SURV	ST ER PROB DENS	ST ER HAZD RATE	MED LIFE EXPECT	ST ER LIFE EXPECT
0.0	0.25	0.50	1000.	31.	0.0310	0.9690	1.0000	0.0620	0.0620	0.0	0.0110	0.0110	4.0636	0.1437
0.5	1.00	1.00	969.	111.	0.1146	0.8854	0.9690	0.1110	0.1146	0.0055	0.0099	0.0102	3.7045	0.1415
1.5	2.00	1.00	858.	160.	0.1865	0.8135	0.8580	0.1600	0.1865	0.0110	0.0116	0.0133	3.2277	0.1450
2.5	3.00	1.00	698.	136.	0.1948	0.8052	0.6980	0.1360	0.1948	0.0145	0.0108	0.0150	3.0250	0.1651
3.5	4.00	1.00	562.	110.	0.1957	0.8043	0.5620	0.1100	0.1957	0.0157	0.0099	0.0167	2.8750	0.1482
4.5	5.00	1.00	452.	101.	0.2235	0.7765	0.4520	0.1010	0.2235	0.0157	0.0095	0.0196	2.6164	0.1456
5.5	6.00	1.00	351.	80.	0.2279	0.7721	0.3510	0.0800	0.2279	0.0151	0.0086	0.0224	2.4327	0.1801
6.5	7.00	1.00	271.	73.	0.2694	0.7306	0.2710	0.0730	0.2694	0.0141	0.0082	0.0269	2.2386	0.1871
7.5	8.00	1.00	198.	52.	0.2626	0.7374	0.1980	0.0520	0.2626	0.0126	0.0070	0.0313	2.1000	0.2345
8.5	9.00	1.00	146.	44.	0.3014	0.6986	0.1460	0.0440	0.3014	0.0112	0.0065	0.0380	1.9667	0.2014
9.5	10.00	1.00	102.	30.	0.2941	0.7059	0.1020	0.0300	0.2941	0.0096	0.0054	0.0451	1.7778	0.1870
10.5	11.00	1.00	72.	27.	0.3750	0.6250	0.0720	0.0270	0.3750	0.0082	0.0051	0.0571	1.5000	0.2357
11.5	12.00	1.00	45.	18.	0.4000	0.6000	0.0450	0.0180	0.4000	0.0066	0.0042	0.0730	1.4091	0.3049
12.5	13.00	1.00	27.	11.	0.4074	0.5926	0.0270	0.0110	0.4074	0.0051	0.0033	0.0946	1.6250	0.6495
13.5	14.00	1.00	16.	4.	0.2500	0.7500	0.0160	0.0040	0.2500	0.0040	0.0020	0.1083	1.5714	0.2857
14.5	15.00	1.00	12.	7.	0.5833	0.4167	0.0120	0.0070	0.5833	0.0034	0.0026	0.1423	0.8571	0.2474
15.5	16.50	2.00	5.	2.	0.4000	0.6000	0.0050	0.0010	0.2000	0.0022	0.0007	0.1095	2.2500	0.5590
17.5	18.00	1.00	3.	2.	0.6667	0.3333	0.0030	0.0020	0.6667	0.0017	0.0014	0.2722	0.7500	0.4330
18.5	**	**	1.	1.	0.5000	0.5000	0.0010	**	**	0.0010	**	**	*	*

* INDICATES NO MEDIAN LIFE EXPECTANCY CAN BE CALCULATED FOR THIS ENTRY.

** CALCULATIONS INVOLVING INTERVAL WIDTH FOR LAST INTERVAL HAVE NO MEANING.

LN-LIKELIHOOD FOR SAMPLE DATA = -3217.07

ESTIMATES OF PARAMETERS

MODEL 1 = EXPONENTIAL
MODEL 2 = LINEAR HAZARD
MODEL 3 = GOMPERTZ
MODEL 4 = WEIBULL

WEIGHT1(I) = 1.
WEIGHT2(I) = 1. / V
WEIGHT3(I) = N(I) * H(I)

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
LAMBDA-0	0.2897	0.1620	0.1888	0.1039	0.0927	0.1058	-2.0580	-2.0290	-2.2058	0.0804	0.0842	0.0830
VAR(LAMBDA-0)	0.0004	0.0000	0.0000	0.0010	0.0001	0.0001	0.0064	0.0024	0.0036	0.0001	0.0000	0.0001
ST.ERROR(LAM-0)	0.0210	0.0050	0.0053	0.0317	0.0071	0.0077	0.0800	0.0491	0.0599	0.0079	0.0062	0.0071
LAMBDA-1				0.0216	0.0244	0.0224	0.0799	0.0930	0.1215	1.4392	1.4227	1.4230
VAR(LAMBDA-1)				0.0000	0.0000	0.0000	0.0002	0.0001	0.0001	0.0025	0.0011	0.0017
ST.ERROR(LAM-1)				0.0059	0.0018	0.0019	0.0135	0.0078	0.0101	0.0504	0.0329	0.0410
COV(LAM-0,LAM-1)							-0.0010	-0.0003	-0.0005	-0.0004	-0.0002	-0.0003
LN-LIKELIHOOD	-2644.70	-2614.68	-2590.15	-2516.40	-2514.62	-2513.17	-2534.23	-2519.22	-2521.38	-2503.34	-2501.95	-2503.49

ESTIMATES OF HAZARD FUNCTION

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
INTERVAL START	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
0.0	0.2897	0.1620	0.1888	0.1093	0.0988	0.1114	0.1303	0.1346	0.1136	0.0629	0.0667	0.0657
0.50	0.2897	0.1620	0.1888	0.1255	0.1171	0.1292	0.1383	0.1443	0.1244	0.1157	0.1198	0.1182
1.50	0.2897	0.1620	0.1888	0.1471	0.1415	0.1505	0.1498	0.1583	0.1405	0.1569	0.1606	0.1584
2.50	0.2897	0.1620	0.1888	0.1687	0.1659	0.1729	0.1623	0.1738	0.1586	0.1874	0.1907	0.1880
3.50	0.2897	0.1620	0.1888	0.1904	0.1903	0.1952	0.1758	0.1907	0.1791	0.2127	0.2153	0.2124
4.50	0.2897	0.1620	0.1888	0.2120	0.2147	0.2176	0.1904	0.2093	0.2022	0.2346	0.2366	0.2334
5.50	0.2897	0.1620	0.1888	0.2336	0.2391	0.2399	0.2063	0.2297	0.2283	0.2541	0.2556	0.2521
6.50	0.2897	0.1620	0.1888	0.2552	0.2635	0.2623	0.2234	0.2520	0.2578	0.2719	0.2728	0.2691
7.50	0.2897	0.1620	0.1888	0.2768	0.2879	0.2846	0.2420	0.2766	0.2911	0.2884	0.2886	0.2847
8.50	0.2897	0.1620	0.1888	0.2984	0.3123	0.3070	0.2622	0.3036	0.3287	0.3037	0.3033	0.2993
9.50	0.2897	0.1620	0.1888	0.3200	0.3367	0.3293	0.2840	0.3331	0.3712	0.3181	0.3172	0.3129
10.50	0.2897	0.1620	0.1888	0.3416	0.3611	0.3517	0.3076	0.3656	0.4191	0.3317	0.3302	0.3258
11.50	0.2897	0.1620	0.1888	0.3633	0.3854	0.3740	0.3332	0.4012	0.4732	0.3446	0.3426	0.3380
12.50	0.2897	0.1620	0.1888	0.3849	0.4098	0.3964	0.3609	0.4403	0.5344	0.3569	0.3544	0.3496
13.50	0.2897	0.1620	0.1888	0.4065	0.4342	0.4187	0.3909	0.4832	0.6034	0.3687	0.3656	0.3608
14.50	0.2897	0.1620	0.1888	0.4281	0.4586	0.4411	0.4235	0.5303	0.6813	0.3801	0.3765	0.3714
15.50	0.2897	0.1620	0.1888	0.4605	0.4952	0.4746	0.4774	0.6097	0.8175	0.3963	0.3919	0.3867
17.50	0.2897	0.1620	0.1888	0.4929	0.5318	0.5082	0.5382	0.7009	0.9809	0.4117	0.4066	0.4012
18.50	**	**	**	**	**	**	**	**	**	**	**	**

ESTIMATES OF SURVIVORSHIP FUNCTION

INTERVAL START	MODEL 1 WT 1	MODEL 1 WT 2	MODEL 1 WT 3	MODEL 2 WT 1	MODEL 2 WT 2	MODEL 2 WT 3	MODEL 3 WT 1	MODEL 3 WT 2	MODEL 3 WT 3	MODEL 4 WT 1	MODEL 4 WT 2	MODEL 4 WT 3
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	0.8651	0.9222	0.9099	0.9468	0.9518	0.9458	0.9369	0.9349	0.9448	0.9708	0.9691	0.9695
1.50	0.6475	0.7843	0.7534	0.8351	0.8466	0.8320	0.8159	0.8093	0.8342	0.8658	0.8608	0.8626
2.50	0.4847	0.6670	0.6238	0.7209	0.7349	0.7158	0.7023	0.6907	0.7249	0.7404	0.7333	0.7365
3.50	0.3628	0.5673	0.5165	0.6089	0.6225	0.6021	0.5971	0.5805	0.6185	0.6140	0.6062	0.6104
4.50	0.2715	0.4824	0.4276	0.5034	0.5146	0.4953	0.5008	0.4797	0.5170	0.4964	0.4888	0.4937
5.50	0.2032	0.4103	0.3541	0.4072	0.4152	0.3985	0.4139	0.3891	0.4223	0.3927	0.3859	0.3909
6.50	0.1521	0.3489	0.2932	0.3224	0.3269	0.3135	0.3368	0.3092	0.3361	0.3046	0.2989	0.3038
7.50	0.1139	0.2968	0.2427	0.2498	0.2512	0.2412	0.2693	0.2403	0.2596	0.2321	0.2275	0.2322
8.50	0.0852	0.2524	0.2010	0.1894	0.1884	0.1814	0.2114	0.1822	0.1940	0.1739	0.1705	0.1747
9.50	0.0638	0.2146	0.1664	0.1405	0.1378	0.1335	0.1626	0.1345	0.1396	0.1284	0.1259	0.1295
10.50	0.0477	0.1825	0.1378	0.1020	0.0984	0.0960	0.1224	0.0964	0.0963	0.0934	0.0917	0.0947
11.50	0.0357	0.1552	0.1141	0.0725	0.0686	0.0675	0.0900	0.0669	0.0633	0.0671	0.0659	0.0684
12.50	0.0267	0.1320	0.0944	0.0504	0.0467	0.0465	0.0645	0.0448	0.0394	0.0475	0.0468	0.0488
13.50	0.0200	0.1123	0.0782	0.0343	0.0310	0.0313	0.0449	0.0288	0.0231	0.0332	0.0328	0.0344
14.50	0.0150	0.0955	0.0647	0.0229	0.0201	0.0206	0.0304	0.0178	0.0126	0.0230	0.0228	0.0240
15.50	0.0112	0.0812	0.0536	0.0149	0.0127	0.0132	0.0199	0.0105	0.0064	0.0157	0.0156	0.0165
17.50	0.0063	0.0587	0.0368	0.0059	0.0047	0.0051	0.0077	0.0031	0.0012	0.0071	0.0071	0.0076
18.50	1.0000											

ESTIMATES OF PROBABILITY DENSITY FUNCTION

INTERVAL START	MODEL 1 WT 1	MODEL 1 WT 2	MODEL 1 WT 3	MODEL 2 WT 1	MODEL 2 WT 2	MODEL 2 WT 3	MODEL 3 WT 1	MODEL 3 WT 2	MODEL 3 WT 3	MODEL 4 WT 1	MODEL 4 WT 2	MODEL 4 WT 3
0.0	0.2897	0.1620	0.1888	0.1093	0.0988	0.1114	0.1303	0.1346	0.1136	0.0629	0.0667	0.0657
0.50	0.2168	0.1378	0.1563	0.1119	0.1059	0.1140	0.1211	0.1257	0.1106	0.1068	0.1101	0.1087
1.50	0.1623	0.1172	0.1294	0.1145	0.1120	0.1165	0.1136	0.1186	0.1095	0.1261	0.1282	0.1268
2.50	0.1215	0.0996	0.1072	0.1121	0.1126	0.1138	0.1053	0.1103	0.1064	0.1268	0.1275	0.1265
3.50	0.0909	0.0847	0.0887	0.1057	0.1080	0.1069	0.0963	0.1009	0.1015	0.1178	0.1175	0.1169
4.50	0.0681	0.0721	0.0735	0.0962	0.0995	0.0969	0.0869	0.0906	0.0948	0.1038	0.1030	0.1028
5.50	0.0509	0.0613	0.0608	0.0849	0.0884	0.0850	0.0772	0.0799	0.0863	0.0881	0.0870	0.0871
6.50	0.0381	0.0521	0.0504	0.0726	0.0757	0.0723	0.0674	0.0689	0.0765	0.0725	0.0713	0.0716
7.50	0.0285	0.0443	0.0417	0.0604	0.0628	0.0597	0.0579	0.0581	0.0656	0.0581	0.0570	0.0574
8.50	0.0214	0.0377	0.0345	0.0488	0.0505	0.0479	0.0487	0.0477	0.0544	0.0455	0.0445	0.0451
9.50	0.0160	0.0321	0.0286	0.0384	0.0393	0.0374	0.0402	0.0381	0.0433	0.0349	0.0341	0.0347
10.50	0.0120	0.0273	0.0237	0.0295	0.0298	0.0284	0.0324	0.0295	0.0329	0.0263	0.0257	0.0263
11.50	0.0090	0.0232	0.0196	0.0220	0.0219	0.0210	0.0255	0.0221	0.0238	0.0195	0.0191	0.0195
12.50	0.0067	0.0197	0.0162	0.0161	0.0156	0.0151	0.0195	0.0159	0.0163	0.0142	0.0139	0.0143
13.50	0.0050	0.0168	0.0134	0.0114	0.0109	0.0106	0.0145	0.0110	0.0104	0.0102	0.0100	0.0104
14.50	0.0039	0.0143	0.0111	0.0079	0.0073	0.0073	0.0105	0.0073	0.0062	0.0072	0.0071	0.0074
15.50	0.0024	0.0112	0.0084	0.0044	0.0039	0.0040	0.0060	0.0036	0.0024	0.0042	0.0042	0.0044
17.50	0.0016	0.0088	0.0063	0.0023	0.0019	0.0020	0.0032	0.0015	0.0008	0.0024	0.0024	0.0025
18.50	**	**	**	**	**	**	**	**	**	**	**	**

V- 1 WEIBULL -- LAMBDA-0 = 0.08 LAMBDA-1 = 1.5

ACTUARIAL ESTIMATE

LIFE TABLE DATA

INT START	MID POINT	INT WIDTH	NO. ENTER	NO. RETIRE	PROPN RETIRE	PROPN SURV	CUM PROPN SURV	PROB DENS	HAZD RATE	ST ER CUM SURV	ST ER PROB DENS	ST ER HAZD RATE	MED LIFE EXPECT	ST ER LIFE EXPECT
0.0	0.25	0.50	1000.	31.	0.0310	0.9690	1.0000	0.0620	0.0630	0.0	0.0110	0.0113	4.0636	0.1437
0.5	1.00	1.00	969.	111.	0.1146	0.8854	0.9690	0.1110	0.1215	0.0055	0.0099	0.0115	3.7045	0.1415
1.5	2.00	1.00	858.	160.	0.1865	0.8135	0.8580	0.1600	0.2057	0.0110	0.0116	0.0162	3.2277	0.1450
2.5	3.00	1.00	698.	136.	0.1948	0.8052	0.6980	0.1360	0.2159	0.0145	0.0108	0.0184	3.0250	0.1651
3.5	4.00	1.00	562.	110.	0.1957	0.8043	0.5620	0.1100	0.2170	0.0157	0.0099	0.0206	2.8750	0.1482
4.5	5.00	1.00	452.	101.	0.2235	0.7765	0.4520	0.1010	0.2516	0.0157	0.0095	0.0248	2.6164	0.1456
5.5	6.00	1.00	351.	80.	0.2279	0.7721	0.3510	0.0800	0.2572	0.0151	0.0086	0.0285	2.4327	0.1801
6.5	7.00	1.00	271.	73.	0.2694	0.7306	0.2710	0.0730	0.3113	0.0141	0.0082	0.0360	2.2386	0.1871
7.5	8.00	1.00	198.	52.	0.2626	0.7374	0.1980	0.0520	0.3023	0.0126	0.0070	0.0414	2.1000	0.2345
8.5	9.00	1.00	146.	44.	0.3014	0.6986	0.1460	0.0440	0.3548	0.0112	0.0065	0.0526	1.9667	0.2014
9.5	10.00	1.00	102.	30.	0.2941	0.7059	0.1020	0.0300	0.3448	0.0096	0.0054	0.0620	1.7778	0.1870
10.5	11.00	1.00	72.	27.	0.3750	0.6250	0.0720	0.0270	0.4615	0.0082	0.0051	0.0864	1.5000	0.2357
11.5	12.00	1.00	45.	18.	0.4000	0.6000	0.0450	0.0180	0.5000	0.0066	0.0042	0.1141	1.4091	0.3049
12.5	13.00	1.00	27.	11.	0.4074	0.5926	0.0270	0.0110	0.5116	0.0051	0.0033	0.1491	1.6250	0.6495
13.5	14.00	1.00	16.	4.	0.2500	0.7500	0.0160	0.0040	0.2857	0.0040	0.0020	0.1414	1.5714	0.2857
14.5	15.00	1.00	12.	7.	0.5833	0.4167	0.0120	0.0070	0.8235	0.0034	0.0026	0.2837	0.8571	0.2474
15.5	16.50	2.00	5.	2.	0.4000	0.6000	0.0050	0.0010	0.2500	0.0022	0.0007	0.1712	2.2500	0.5590
17.5	18.00	1.00	3.	2.	0.6667	0.3333	0.0030	0.0020	1.0000	0.0017	0.0014	0.6124	0.7500	0.4330
18.5	**	**	1.	1.	0.5000	0.5000	0.0010	**	**	0.0010	**	**	*	*

* INDICATES NO MEDIAN LIFE EXPECTANCY CAN BE CALCULATED FOR THIS ENTRY.
** CALCULATIONS INVOLVING INTERVAL WIDTH FOR LAST INTERVAL HAVE NO MEANING.

LN-LIKELIHOOD FOR SAMPLE DATA = -3217.07

ESTIMATES OF PARAMETERS

MODEL 1 = EXPONENTIAL
MODEL 2 = LINEAR HAZARD
MODEL 3 = GOMPERTZ
MODEL 4 = WEIBULL

WEIGHT1(I) = 1.
WEIGHT2(I) = 1. / V
WEIGHT3(I) = N(I) * H(I)

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
LAMBDA-0	0.3599	0.1638	0.2122	0.0774	0.0082	0.1063	-2.0232	-1.9949	-2.1571	0.0807	0.0868	0.0860
VAR(LAMBDA-0)	0.0017	0.0000	0.0000	0.0046	0.0001	0.0001	0.0103	0.0030	0.0040	0.0001	0.0000	0.0001
ST.ERROR(LAM-0)	0.0416	0.0058	0.0067	0.0678	0.0080	0.0098	0.1013	0.0549	0.0632	0.0088	0.0068	0.0076
LAMBDA-1				0.0329	0.0315	0.0285	0.0952	0.1090	0.1349	1.5049	1.4644	1.4632
VAR(LAMBDA-1)				0.0002	0.0000	0.0000	0.0003	0.0001	0.0001	0.0038	0.0013	0.0018
ST.ERROR(LAM-1)				0.0124	0.0023	0.0027	0.0178	0.0093	0.0109	0.0619	0.0364	0.0426
COV(LAM-0,LAM-1)							-0.0017	-0.0004	-0.0006	-0.0005	-0.0002	-0.0003
LN-LIKELIHOOD	-2762.08	-2612.21	-2585.33	-2506.44	-2503.98	-2503.91	-2517.47	-2513.10	-2515.76	-2496.69	-2495.68	-2495.75

ESTIMATES OF HAZARD FUNCTION

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
INTERVAL START	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
0.0	0.3599	0.1638	0.2122	0.0856	0.0961	0.1134	0.1354	0.1398	0.1196	0.0603	0.0668	0.0662
0.50	0.3599	0.1638	0.2122	0.1102	0.1197	0.1348	0.1454	0.1517	0.1324	0.1214	0.1271	0.1258
1.50	0.3599	0.1638	0.2122	0.1431	0.1511	0.1634	0.1600	0.1692	0.1515	0.1723	0.1754	0.1734
2.50	0.3599	0.1638	0.2122	0.1759	0.1826	0.1919	0.1759	0.1887	0.1734	0.2115	0.2118	0.2092
3.50	0.3599	0.1638	0.2122	0.2088	0.2141	0.2205	0.1935	0.2104	0.1984	0.2446	0.2420	0.2390
4.50	0.3599	0.1638	0.2122	0.2417	0.2455	0.2490	0.2128	0.2346	0.2271	0.2737	0.2685	0.2651
5.50	0.3599	0.1638	0.2122	0.2745	0.2770	0.2776	0.2341	0.2617	0.2599	0.3001	0.2922	0.2884
6.50	0.3599	0.1638	0.2122	0.3074	0.3085	0.3061	0.2575	0.2918	0.2975	0.3244	0.3139	0.3098
7.50	0.3599	0.1638	0.2122	0.3402	0.3400	0.3347	0.2832	0.3254	0.3404	0.3470	0.3339	0.3295
8.50	0.3599	0.1638	0.2122	0.3731	0.3714	0.3632	0.3115	0.3629	0.3896	0.3683	0.3527	0.3480
9.50	0.3599	0.1638	0.2122	0.4060	0.4029	0.3918	0.3426	0.4047	0.4459	0.3884	0.3704	0.3654
10.50	0.3599	0.1638	0.2122	0.4388	0.4344	0.4203	0.3768	0.4513	0.5103	0.4076	0.3872	0.3819
11.50	0.3599	0.1638	0.2122	0.4717	0.4658	0.4489	0.4144	0.5033	0.5841	0.4259	0.4031	0.3976
12.50	0.3599	0.1638	0.2122	0.5045	0.4973	0.4774	0.4558	0.5613	0.6684	0.4435	0.4184	0.4126
13.50	0.3599	0.1638	0.2122	0.5374	0.5288	0.5060	0.5013	0.6259	0.7650	0.4604	0.4331	0.4270
14.50	0.3599	0.1638	0.2122	0.5702	0.5602	0.5345	0.5514	0.6981	0.8755	0.4767	0.4472	0.4409
15.50	0.3599	0.1638	0.2122	0.6195	0.6074	0.5773	0.6360	0.8221	1.0720	0.5002	0.4674	0.4608
17.50	0.3599	0.1638	0.2122	0.6688	0.6546	0.6202	0.7336	0.9682	1.3125	0.5227	0.4867	0.4797
18.50	**	**	**	**	**	**	**	**	**	**	**	**

ESTIMATES OF SURVIVORSHIP FUNCTION

INTERVAL START	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	0.8353	0.9214	0.8993	0.9581	0.9531	0.9449	0.9345	0.9325	0.9419	0.9720	0.9690	0.9693
1.50	0.5829	0.7821	0.7274	0.8581	0.8456	0.8257	0.8080	0.8012	0.8251	0.8620	0.8545	0.8559
2.50	0.4067	0.6639	0.5883	0.7437	0.7270	0.7013	0.6885	0.6764	0.7090	0.7258	0.7174	0.7200
3.50	0.2838	0.5636	0.4758	0.6237	0.6057	0.5788	0.5774	0.5601	0.5961	0.5876	0.5806	0.5842
4.50	0.1980	0.4785	0.3848	0.5062	0.4889	0.4643	0.4758	0.4538	0.4887	0.4602	0.4559	0.4601
5.50	0.1382	0.4062	0.3113	0.3975	0.3825	0.3620	0.3845	0.3588	0.3894	0.3500	0.3486	0.3530
6.50	0.0964	0.3448	0.2517	0.3021	0.2899	0.2742	0.3043	0.2762	0.3002	0.2593	0.2603	0.2646
7.50	0.0673	0.2927	0.2036	0.2221	0.2130	0.2019	0.2352	0.2063	0.2229	0.1875	0.1902	0.1941
8.50	0.0469	0.2485	0.1647	0.1581	0.1516	0.1445	0.1772	0.1489	0.1585	0.1325	0.1362	0.1396
9.50	0.0328	0.2109	0.1332	0.1089	0.1046	0.1005	0.1297	0.1036	0.1074	0.0917	0.0957	0.0986
10.50	0.0229	0.1790	0.1077	0.0725	0.0699	0.0679	0.0921	0.0691	0.0687	0.0622	0.0661	0.0684
11.50	0.0159	0.1520	0.0871	0.0468	0.0453	0.0446	0.0632	0.0440	0.0412	0.0414	0.0449	0.0467
12.50	0.0111	0.1290	0.0705	0.0292	0.0284	0.0285	0.0417	0.0266	0.0230	0.0270	0.0300	0.0314
13.50	0.0078	0.1095	0.0570	0.0176	0.0173	0.0177	0.0265	0.0152	0.0118	0.0173	0.0197	0.0208
14.50	0.0054	0.0930	0.0461	0.0103	0.0102	0.0107	0.0160	0.0081	0.0055	0.0109	0.0128	0.0136
15.50	0.0038	0.0789	0.0373	0.0058	0.0058	0.0062	0.0092	0.0040	0.0023	0.0068	0.0082	0.0087
17.50	0.0018	0.0569	0.0244	0.0017	0.0017	0.0020	0.0026	0.0008	0.0003	0.0025	0.0032	0.0035
18.50	1.0000											

ESTIMATES OF PROBABILITY DENSITY FUNCTION

INTERVAL START	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
0.0	0.3599	0.1638	0.2122	0.0856	0.0961	0.1134	0.1354	0.1398	0.1196	0.0603	0.0668	0.0662
0.50	0.2511	0.1391	0.1716	0.1004	0.1079	0.1195	0.1266	0.1314	0.1170	0.1120	0.1166	0.1154
1.50	0.1752	0.1181	0.1388	0.1148	0.1190	0.1247	0.1195	0.1248	0.1162	0.1371	0.1380	0.1368
2.50	0.1223	0.1002	0.1123	0.1203	0.1216	0.1227	0.1112	0.1164	0.1130	0.1387	0.1372	0.1362
3.50	0.0853	0.0851	0.0908	0.1178	0.1170	0.1147	0.1017	0.1064	0.1075	0.1277	0.1250	0.1244
4.50	0.0595	0.0722	0.0734	0.1088	0.1066	0.1024	0.0913	0.0950	0.0994	0.1102	0.1074	0.1071
5.50	0.0415	0.0613	0.0594	0.0955	0.0926	0.0878	0.0803	0.0827	0.0892	0.0907	0.0883	0.0884
6.50	0.0290	0.0520	0.0480	0.0800	0.0770	0.0723	0.0691	0.0699	0.0773	0.0717	0.0700	0.0704
7.50	0.0202	0.0442	0.0389	0.0640	0.0613	0.0574	0.0580	0.0573	0.0644	0.0549	0.0539	0.0544
8.50	0.0141	0.0375	0.0314	0.0491	0.0469	0.0439	0.0474	0.0453	0.0512	0.0407	0.0404	0.0409
9.50	0.0098	0.0318	0.0254	0.0362	0.0346	0.0325	0.0376	0.0344	0.0386	0.0294	0.0295	0.0301
10.50	0.0069	0.0270	0.0206	0.0257	0.0245	0.0232	0.0289	0.0250	0.0274	0.0207	0.0211	0.0216
11.50	0.0048	0.0229	0.0166	0.0175	0.0168	0.0161	0.0214	0.0173	0.0182	0.0143	0.0148	0.0153
12.50	0.0033	0.0195	0.0134	0.0115	0.0111	0.0107	0.0152	0.0114	0.0111	0.0096	0.0102	0.0106
13.50	0.0023	0.0165	0.0109	0.0073	0.0070	0.0070	0.0104	0.0070	0.0062	0.0064	0.0069	0.0072
14.50	0.0016	0.0140	0.0088	0.0044	0.0043	0.0044	0.0067	0.0040	0.0031	0.0041	0.0046	0.0048
15.50	0.0009	0.0110	0.0064	0.0020	0.0020	0.0021	0.0032	0.0015	0.0009	0.0021	0.0024	0.0026
17.50	0.0006	0.0086	0.0047	0.0008	0.0008	0.0009	0.0013	0.0005	0.0002	0.0010	0.0012	0.0013
18.50	**	**	**	**	**	**	**	**	**	**	**	**

V- 1 WEIBULL -- LAMBDA-0 = 0.08 LAMBDA-1 = 1.5

MAXIMUM LIKELIHOOD ESTIMATE

LIFE TABLE DATA

INT START	MID POINT	INT WIDTH	NO. ENTER	NO. RETIRE	PROPN RETIRE	PROPN SURV	CUM PROPN SURV	PROB DENS	HAZD RATE	ST ER CUM SURV	ST ER PROB DENS	ST ER HAZD RATE	MED LIFE EXPECT	ST ER LIFE EXPECT
0.0	0.25	0.50	1000.	31.	0.0310	0.9690	1.0000	0.0620	0.0630	0.0	0.0110	0.0113	4.0636	0.1437
0.5	1.00	1.00	969.	111.	0.1146	0.8854	0.9690	0.1110	0.1217	0.0055	0.0099	0.0116	3.7045	0.1415
1.5	2.00	1.00	858.	160.	0.1865	0.8135	0.8580	0.1600	0.2064	0.0110	0.0116	0.0163	3.2277	0.1450
2.5	3.00	1.00	698.	136.	0.1948	0.8052	0.6980	0.1360	0.2167	0.0145	0.0108	0.0186	3.0250	0.1651
3.5	4.00	1.00	562.	110.	0.1957	0.8043	0.5620	0.1100	0.2178	0.0157	0.0099	0.0208	2.8750	0.1482
4.5	5.00	1.00	452.	101.	0.2235	0.7765	0.4520	0.1010	0.2529	0.0157	0.0095	0.0252	2.6164	0.1456
5.5	6.00	1.00	351.	80.	0.2279	0.7721	0.3510	0.0800	0.2587	0.0151	0.0086	0.0290	2.4327	0.1801
6.5	7.00	1.00	271.	73.	0.2694	0.7306	0.2710	0.0730	0.3139	0.0141	0.0082	0.0369	2.2386	0.1871
7.5	8.00	1.00	198.	52.	0.2626	0.7374	0.1980	0.0520	0.3047	0.0126	0.0070	0.0424	2.1000	0.2345
8.5	9.00	1.00	146.	44.	0.3014	0.6986	0.1460	0.0440	0.3586	0.0112	0.0065	0.0544	1.9667	0.2014
9.5	10.00	1.00	102.	30.	0.2941	0.7059	0.1020	0.0300	0.3483	0.0096	0.0054	0.0639	1.7778	0.1870
10.5	11.00	1.00	72.	27.	0.3750	0.6250	0.0720	0.0270	0.4700	0.0082	0.0051	0.0913	1.5000	0.2357
11.5	12.00	1.00	45.	18.	0.4000	0.6000	0.0450	0.0180	0.5108	0.0066	0.0042	0.1217	1.4091	0.3049
12.5	13.00	1.00	27.	11.	0.4074	0.5926	0.0270	0.0110	0.5232	0.0051	0.0033	0.1596	1.6250	0.6495
13.5	14.00	1.00	16.	4.	0.2500	0.7500	0.0160	0.0040	0.2877	0.0040	0.0020	0.1443	1.5714	0.2857
14.5	15.00	1.00	12.	7.	0.5833	0.4167	0.0120	0.0070	0.8755	0.0034	0.0026	0.3416	0.8571	0.2474
15.5	16.50	2.00	5.	2.	0.4000	0.6000	0.0050	0.0010	0.2554	0.0022	0.0007	0.1826	2.2500	0.5590
17.5	18.00	1.00	3.	2.	0.6667	0.3333	0.0030	0.0020	1.0986	0.0017	0.0014	0.8165	0.7500	0.4330
18.5	**	**	1.	1.	0.5000	0.5000	0.0010	**	**	0.0010	**	**	*	*

* INDICATES NO MEDIAN LIFE EXPECTANCY CAN BE CALCULATED FOR THIS ENTRY.

** CALCULATIONS INVOLVING INTERVAL WIDTH FOR LAST INTERVAL HAVE NO MEANING.

LN-LIKELIHOOD FOR SAMPLE DATA = -3217.07

ESTIMATES OF PARAMETERS

MODEL 1 = EXPONENTIAL
MODEL 2 = LINEAR HAZARD
MODEL 3 = GOMPERTZ
MODEL 4 = WEIBULL

WEIGHT1(I) = 1.
WEIGHT2(I) = 1. / V
WEIGHT3(I) = N(I) * H(I)

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
LAMBDA-0	0.3713	0.1630	0.2136	0.0651	0.0875	0.1051	-2.0333	-1.9986	-2.1576	0.0799	0.0870	0.0859
VAR(LAMBDA-0)	0.0028	0.0000	0.0000	0.0079	0.0001	0.0001	0.0126	0.0031	0.0040	0.0001	0.0000	0.0001
ST.ERROR(LAM-0)	0.0527	0.0059	0.0068	0.0889	0.0081	0.0102	0.1121	0.0557	0.0635	0.0091	0.0068	0.0076
LAMBDA-1				0.0356	0.0320	0.0292	0.0983	0.1110	0.1363	1.5160	1.4654	1.4664
VAR(LAMBDA-1)				0.0003	0.0000	0.0000	0.0004	0.0001	0.0001	0.0045	0.0014	0.0018
ST.ERROR(LAM-1)				0.0162	0.0024	0.0029	0.0198	0.0096	0.0110	0.0672	0.0369	0.0427
COV(LAM-0,LAM-1)							-0.0021	-0.0004	-0.0006	-0.0006	-0.0002	-0.0003
LN-LIKELIHOOD	-2786.23	-2613.31	-2585.46	-2510.34	-2503.03	-2503.68	-2516.52	-2513.13	-2516.06	-2497.30	-2495.70	-2495.73

ESTIMATES OF HAZARD FUNCTION

	MODEL 1			MODEL 2			MODEL 3			MODEL 4		
INTERVAL START	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3	WT 1	WT 2	WT 3
0.0	0.3713	0.1630	0.2136	0.0740	0.0955	0.1124	0.1342	0.1393	0.1196	0.0592	0.0669	0.0660
0.50	0.3713	0.1630	0.2136	0.1007	0.1195	0.1343	0.1444	0.1514	0.1325	0.1211	0.1275	0.1260
1.50	0.3713	0.1630	0.2136	0.1363	0.1515	0.1636	0.1594	0.1692	0.1518	0.1732	0.1760	0.1741
2.50	0.3713	0.1630	0.2136	0.1719	0.1035	0.1928	0.1758	0.1891	0.1740	0.2135	0.2126	0.2103
3.50	0.3713	0.1630	0.2136	0.2076	0.2155	0.2221	0.1940	0.2113	0.1994	0.2476	0.2431	0.2405
4.50	0.3713	0.1630	0.2136	0.2432	0.2475	0.2513	0.2140	0.2361	0.2285	0.2779	0.2697	0.2669
5.50	0.3713	0.1630	0.2136	0.2788	0.2795	0.2805	0.2362	0.2638	0.2619	0.3053	0.2935	0.2906
6.50	0.3713	0.1630	0.2136	0.3144	0.3114	0.3098	0.2606	0.2947	0.3002	0.3305	0.3154	0.3122
7.50	0.3713	0.1630	0.2136	0.3500	0.3434	0.3390	0.2875	0.3293	0.3440	0.3541	0.3356	0.3323
8.50	0.3713	0.1630	0.2136	0.3857	0.3754	0.3683	0.3172	0.3680	0.3942	0.3763	0.3545	0.3510
9.50	0.3713	0.1630	0.2136	0.4213	0.4074	0.3975	0.3500	0.4112	0.4518	0.3973	0.3723	0.3687
10.50	0.3713	0.1630	0.2136	0.4569	0.4394	0.4268	0.3861	0.4594	0.5178	0.4174	0.3892	0.3855
11.50	0.3713	0.1630	0.2136	0.4925	0.4714	0.4560	0.4260	0.5133	0.5934	0.4365	0.4053	0.4014
12.50	0.3713	0.1630	0.2136	0.5282	0.5034	0.4852	0.4701	0.5736	0.6801	0.4549	0.4207	0.4167
13.50	0.3713	0.1630	0.2136	0.5638	0.5354	0.5145	0.5186	0.6409	0.7794	0.4727	0.4354	0.4314
14.50	0.3713	0.1630	0.2136	0.5994	0.5674	0.5437	0.5722	0.7161	0.8932	0.4898	0.4496	0.4455
15.50	0.3713	0.1630	0.2136	0.6529	0.6153	0.5876	0.6632	0.8459	1.0958	0.5145	0.4700	0.4657
17.50	0.3713	0.1630	0.2136	0.7063	0.6633	0.6315	0.7686	0.9991	1.3444	0.5381	0.4895	0.4850
18.50	**	**	**	**	**	**	**	**	**	**	**	**

ESTIMATES OF SURVIVORSHIP FUNCTION

INTERVAL START	MODEL 1 WT 1	MODEL 1 WT 2	MODEL 1 WT 3	MODEL 2 WT 1	MODEL 2 WT 2	MODEL 2 WT 3	MODEL 3 WT 1	MODEL 3 WT 2	MODEL 3 WT 3	MODEL 4 WT 1	MODEL 4 WT 2	MODEL 4 WT 3
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	0.8306	0.9217	0.8987	0.9637	0.9534	0.9454	0.9351	0.9327	0.9419	0.9725	0.9690	0.9694
1.50	0.5729	0.7831	0.7258	0.8714	0.8460	0.8265	0.8093	0.8016	0.8250	0.8627	0.8542	0.8558
2.50	0.3952	0.6653	0.5862	0.7604	0.7270	0.7018	0.6900	0.6767	0.7087	0.7258	0.7166	0.7194
3.50	0.2726	0.5653	0.4735	0.6402	0.6052	0.5787	0.5787	0.5601	0.5954	0.5865	0.5795	0.5831
4.50	0.1881	0.4803	0.3824	0.5202	0.4878	0.4635	0.4767	0.4534	0.4877	0.4579	0.4546	0.4585
5.50	0.1297	0.4080	0.3089	0.4079	0.3809	0.3605	0.3848	0.3580	0.3880	0.3468	0.3472	0.3512
6.50	0.0895	0.3467	0.2495	0.3087	0.2880	0.2723	0.3038	0.2750	0.2985	0.2556	0.2589	0.2626
7.50	0.0617	0.2945	0.2015	0.2254	0.2110	0.1998	0.2341	0.2048	0.2211	0.1837	0.1889	0.1922
8.50	0.0426	0.2502	0.1627	0.1588	0.1496	0.1423	0.1756	0.1473	0.1567	0.1289	0.1350	0.1379
9.50	0.0294	0.2126	0.1314	0.1080	0.1028	0.0985	0.1278	0.1019	0.1056	0.0885	0.0947	0.0971
10.50	0.0203	0.1806	0.1062	0.0709	0.0684	0.0662	0.0901	0.0675	0.0672	0.0595	0.0653	0.0671
11.50	0.0140	0.1535	0.0857	0.0449	0.0441	0.0432	0.0612	0.0427	0.0400	0.0392	0.0442	0.0457
12.50	0.0096	0.1304	0.0692	0.0274	0.0275	0.0274	0.0400	0.0255	0.0221	0.0253	0.0295	0.0306
13.50	0.0067	0.1108	0.0559	0.0162	0.0166	0.0168	0.0250	0.0144	0.0112	0.0161	0.0194	0.0202
14.50	0.0046	0.0941	0.0452	0.0092	0.0097	0.0101	0.0149	0.0076	0.0051	0.0100	0.0125	0.0131
15.50	0.0032	0.0800	0.0365	0.0051	0.0055	0.0058	0.0084	0.0037	0.0021	0.0061	0.0080	0.0084
17.50	0.0015	0.0577	0.0238	0.0014	0.0016	0.0018	0.0022	0.0007	0.0002	0.0022	0.0031	0.0033
18.50	1.0000											

ESTIMATES OF PROBABILITY DENSITY FUNCTION

INTERVAL START	MODEL 1 WT 1	MODEL 1 WT 2	MODEL 1 WT 3	MODEL 2 WT 1	MODEL 2 WT 2	MODEL 2 WT 3	MODEL 3 WT 1	MODEL 3 WT 2	MODEL 3 WT 3	MODEL 4 WT 1	MODEL 4 WT 2	MODEL 4 WT 3
0.0	0.3713	0.1630	0.2136	0.0740	0.0955	0.1124	0.1342	0.1393	0.1196	0.0592	0.0669	0.0660
0.50	0.2561	0.1385	0.1725	0.0927	0.1078	0.1192	0.1259	0.1312	0.1170	0.1118	0.1169	0.1156
1.50	0.1767	0.1176	0.1393	0.1114	0.1193	0.1250	0.1193	0.1249	0.1164	0.1378	0.1384	0.1373
2.50	0.1219	0.1000	0.1125	0.1205	0.1222	0.1233	0.1114	0.1167	0.1134	0.1390	0.1376	0.1368
3.50	0.0841	0.0849	0.0909	0.1203	0.1175	0.1154	0.1021	0.1068	0.1078	0.1280	0.1252	0.1248
4.50	0.0580	0.0721	0.0734	0.1125	0.1071	0.1031	0.0919	0.0954	0.0998	0.1111	0.1075	0.1074
5.50	0.0400	0.0613	0.0593	0.0994	0.0929	0.0882	0.0810	0.0831	0.0895	0.0912	0.0882	0.0885
6.50	0.0276	0.0521	0.0479	0.0833	0.0771	0.0725	0.0697	0.0702	0.0775	0.0718	0.0699	0.0703
7.50	0.0190	0.0442	0.0387	0.0665	0.0613	0.0574	0.0585	0.0574	0.0644	0.0546	0.0537	0.0542
8.50	0.0131	0.0376	0.0312	0.0507	0.0467	0.0438	0.0477	0.0453	0.0511	0.0403	0.0402	0.0407
9.50	0.0091	0.0319	0.0252	0.0370	0.0343	0.0322	0.0377	0.0343	0.0384	0.0289	0.0293	0.0298
10.50	0.0062	0.0271	0.0204	0.0259	0.0242	0.0229	0.0288	0.0248	0.0271	0.0202	0.0210	0.0214
11.50	0.0043	0.0231	0.0165	0.0174	0.0165	0.0157	0.0212	0.0171	0.0178	0.0138	0.0147	0.0150
12.50	0.0030	0.0196	0.0133	0.0112	0.0108	0.0105	0.0149	0.0111	0.0108	0.0092	0.0101	0.0104
13.50	0.0021	0.0166	0.0107	0.0069	0.0068	0.0067	0.0101	0.0067	0.0060	0.0060	0.0068	0.0070
14.50	0.0014	0.0141	0.0087	0.0041	0.0042	0.0042	0.0064	0.0038	0.0030	0.0038	0.0045	0.0047
15.50	0.0008	0.0111	0.0063	0.0017	0.0019	0.0019	0.0030	0.0014	0.0008	0.0019	0.0024	0.0025
17.50	0.0005	0.0087	0.0046	0.0007	0.0008	0.0008	0.0012	0.0004	0.0002	0.0009	0.0012	0.0013
18.50	**	**	**	**	**	**	**	**	**	**	**	**